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Top-degree components of Grothendieck and Lascoux polynomials

Jianping Pan & Tianyi Yu

Abstract The Castelnuovo–Mumford polynomial $\widehat{\mathfrak{G}}_w$ with $w \in S_n$ is the highest homogeneous component of the Grothendieck polynomial $\mathfrak{G}_w$. Pechenik, Speyer and Weigandt define a statistic $\mathsf{rajcode}(\cdot)$ on S_n that gives the leading monomial of $\widehat{\mathfrak{G}}_w$. We introduce a statistic $\mathsf{rajcode}(\cdot)$ on any diagram D through a combinatorial construction "snow diagram" that augments and decorates D. When D is the Rothe diagram of a permutation w, $\mathsf{rajcode}(D)$ agrees with the aforementioned $\mathsf{rajcode}(w)$. When D is the key diagram of a weak composition α, $\mathsf{rajcode}(D)$ yields the leading monomial of $\widehat{\mathfrak{L}}_\alpha$, the highest homogeneous component of the Lascoux polynomials $\mathfrak{L}_\alpha$. We use $\widehat{\mathfrak{L}}_\alpha$ to construct a basis of $\widehat{V}_n$, the span of $\widehat{\mathfrak{G}}_w$ with $w \in S_n$. Then we show $\widehat{V}_n$ gives a natural algebraic interpretation of a classical q-analogue of Bell numbers.

1. Introduction

Introduced by Lascoux and Schützenberger [13], the Grothendieck polynomial $\mathfrak{G}_w$ is a polynomial representative of the K-class of structure sheaves of Schubert varieties of flag varieties. It is the inhomogeneous analogue of the Schubert polynomial $\mathfrak{S}_w$: The lowest-degree component of $\mathfrak{G}_w$ forms $\mathfrak{S}_w$. Pechenik, Speyer and Weigandt [18] introduce the Castelnuovo–Mumford polynomial $\widehat{\mathfrak{G}}_w$[(1)], the top-degree component of $\mathfrak{G}_w$. They describe the leading monomial of $\widehat{\mathfrak{G}}_w$ with respect to the tail lexicographic order by defining a new statistic $\mathsf{rajcode}(\cdot)$ on S_n. We summarize some of their results on $\widehat{\mathfrak{G}}_w$.

Theorem 1.1 ([18]). *Let w, u be permutations in S_n.*

- (A) *The polynomial $\widehat{\mathfrak{G}}_w$ has leading monomial $x^{\mathsf{rajcode}(w)}$.*
- (B) *We have $\widehat{\mathfrak{G}}_w$ is a scalar multiple of $\widehat{\mathfrak{G}}_u$ if and only if $\mathsf{rajcode}(w) = \mathsf{rajcode}(u)$.*
- (C) *If w is inverse fireworks (see §5), then $x^{\mathsf{rajcode}(w)}$ has coefficient 1 in $\widehat{\mathfrak{G}}_w$. Moreover, there exists exactly one $u' \in S_n$ that is inverse fireworks such that $\mathsf{rajcode}(u) = \mathsf{rajcode}(u')$.*

Dreyer, Mészáros and St. Dizier [6] provide an alternative proof of (A) via the climbing chain model for Grothendieck polynomials introduced by Lenart, Robinson, and Sottile [15]. Hafner [9] provides an alternative proof of (A) for vexillary permutations via bumpless pipedreams.

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(1) Pechenik, Speyer and Weigandt [18] denote it as $\mathfrak{CM}_w$.

Schubert polynomials are related to key polynomials κ_α which are indexed by weak compositions. The key polynomials are the characters of Demazure modules [5]. Both Schubert and key polynomials can be defined recursively via the divided difference operators (see §2). In addition, Schubert polynomials expand positively into key polynomials [19]. The key polynomials also have inhomogeneous analogues called Lascoux polynomials $\mathfrak{L}_\alpha$ [12]. Grothendieck polynomials and Lascoux polynomials are related: An expansion of Grothendieck polynomials into Lascoux polynomials was conjectured by Reiner and Yong [20] and proven by Shimozono and Yu [24].

Due to the connection between $\mathfrak{G}_w$ and $\mathfrak{L}_\alpha$, one would expect the top Lascoux polynomial $\widehat{\mathfrak{L}}_\alpha$, the top-degree component of $\mathfrak{L}_\alpha$, to parallel $\widehat{\mathfrak{G}}_w$. We define a statistic $\mathsf{rajcode}(\cdot)$ on weak compositions and show in §4 that $\widehat{\mathfrak{L}}_\alpha$ enjoy properties analogous to the properties of $\widehat{\mathfrak{G}}_w$ listed in Theorem 1.1:

Theorem 1.2. *Let α and γ be two weak compositions.*

(a) *The polynomial $\widehat{\mathfrak{L}}_\alpha$ has leading monomial $x^{\mathsf{rajcode}(\alpha)}$.*
(b) *We have $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{L}}_\gamma$ if and only if $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$.*
(c) *We say α is snowy if its positive entries are distinct. If α is snowy, then $x^{\mathsf{rajcode}(\alpha)}$ has coefficient 1 in $\widehat{\mathfrak{L}}_\alpha$. Moreover, there exists exactly one snowy weak composition γ' such that $\mathsf{rajcode}(\gamma) = \mathsf{rajcode}(\gamma')$.*

Our definition of $\mathsf{rajcode}(\cdot)$ on weak compositions is diagrammatic. Given a diagram D, we define a combinatorial construction called the snow diagram that augments and decorates D. Let $\mathsf{rajcode}(D)$ be the weight of the snow diagram. Every weak composition α is naturally associated with a diagram called the key diagram $D(\alpha)$ (see Subsection 2.2). Then we define $\mathsf{rajcode}(\alpha) := \mathsf{rajcode}(D(\alpha))$.

Snow diagrams unify the computation of leading monomials in $\widehat{\mathfrak{G}}_w$ and $\widehat{\mathfrak{L}}_\alpha$. Each permutation w is also associated with a diagram called the Rothe diagram $RD(w)$. In §5, we show $\mathsf{rajcode}(w) = \mathsf{rajcode}(RD(w))$. In other words, we give a diagrammatic way to compute $\mathsf{rajcode}(w)$.

Finally, let $\widehat{V}_n := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{G}}_w : w \in S_n\}$ and $\widehat{V} := \bigcup_{n \geqslant 1} \widehat{V}_n$. In Proposition 2.7, we show $\widehat{V}$ is a filtered algebra. Theorem 1.1 can be used to construct a basis of $\widehat{V}_n$ and $\widehat{V}$ consisting of $\widehat{\mathfrak{G}}_w$. In particular, the dimension of $\widehat{V}_n$ is B_n, the n^{th} Bell number. In §6, we use Theorem 1.2 to construct another basis consisting of $\widehat{\mathfrak{L}}_\alpha$. This basis allows us to compute the Hilbert series of $\widehat{V}_n$ and $\widehat{V}$ involving a q-analogue of B_n.

The rest of the paper is organized as follows. In §2, we provide necessary background information and notation. In §3, we construct a snow diagram from any diagram and define statistics $\mathsf{rajcode}(\cdot)$ and $\mathsf{raj}(\cdot)$ on all diagrams. In §4, we prove Theorem 1.2. In §5, we show the statistics $\mathsf{rajcode}(\cdot)$ and $\mathsf{raj}(\cdot)$ on a Rothe diagram are equivalent to that defined in [18]. We also relate the snow diagram to two classical constructions: Schensted insertion and the shadow diagram. In §6, we derive the Hilbert series of $\widehat{V}_n$ and $\widehat{V}$. In §7, we present several open problems and future directions.

2. Background

2.1. Polynomials.

We provide necessary background for Grothendieck polynomials and Lascoux polynomials. Then we introduce $\widehat{\mathfrak{G}}_w$ and $\widehat{\mathfrak{L}}_\alpha$ which span the spaces $\widehat{V}_n$ and $\widehat{V}$.

The *Grothendieck polynomials* $\mathfrak{G}_w \in \mathbb{Z}_{\geqslant 0}[x_1, x_2, \dots][\beta]$ were recursively defined by Lascoux and Schützenberger [13]. Let $\partial_i(\cdot)$ be the divided difference operators acting on the polynomial ring. For each i, define $\partial_i(f) := \dfrac{f - s_i f}{x_i - x_{i+1}}$, where s_i is the operator that swaps x_i and x_{i+1}. Then for $w \in S_n$,

$$\mathfrak{G}_w := \begin{cases} x_1^{n-1}x_2^{n-2}\cdots x_{n-1} & \text{if } w \text{ is } [n, n-1, \ldots, 1] \text{ in one-line notation,} \\ \partial_i((1+\beta x_{i+1})\mathfrak{G}_{ws_i}) & \text{if } w(i) < w(i+1). \end{cases}$$

Let S_+ be the set of permutations of $\{1, 2, \ldots\}$ such that only finitely many numbers are permuted. Take $w \in S_+$ and assume w only permutes numbers in $[n]$. Let $w' \in S_n$ be the restriction of w to $[n]$ and define $\mathfrak{G}_w$ as $\mathfrak{G}_{w'}$. It is shown in [13] that $\mathfrak{G}_w$ is well-defined.

A *weak composition* is an infinite sequence of non-negative integers with finitely many positive entries. Let C_+ be the set of weak compositions. For $\alpha \in C_+$, we use α_i to denote its i^{th} entry, and write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where α_n is the last positive entry. We use x^α to denote the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $|\alpha| = \sum_{i\geqslant 1}^{n} \alpha_i$. The *Lascoux polynomials* $\mathfrak{L}_\alpha$, indexed by weak compositions, are in $\mathbb{Z}_{\geqslant 0}[x_1, x_2, \ldots][\beta]$. By [12], they are defined recursively

$$\mathfrak{L}_\alpha = \begin{cases} x^\alpha & \text{if } \alpha \text{ is weakly decreasing,} \\ \pi_i((1+\beta x_{i+1})\mathfrak{L}_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}, \end{cases}$$

where π_i is the operator $\pi_i(f) := \partial_i(x_i f)$.

We say a pair (i, j) is an *inversion* of $w \in S_n$ if $i < j$ and $w(i) > w(j)$. Let $\mathsf{Inv}(w)$ be the set of all inversions in w and let $\mathsf{inv}(w) = |\mathsf{Inv}(w)|$. Then we may view $\mathfrak{G}_w$ as a polynomial in β, where

$$[\beta^d]\mathfrak{G}_w \quad := \quad \text{coefficient of } \beta^d \text{ in } \mathfrak{G}_w$$

is a homogeneous polynomial in the x-variables with degree $\mathsf{inv}(w) + d$ in $\mathbb{Z}_{\geqslant 0}[x_1, x_2, \ldots]$. The *Schubert polynomial* $\mathfrak{S}_w := [\beta^0]\mathfrak{G}_w$. Similarly, viewing $\mathfrak{L}_\alpha$ as a polynomial of β, $[\beta^d]\mathfrak{L}_\alpha$ is a homogeneous polynomial with degree $|\alpha| + d$ in $\mathbb{Z}_{\geqslant 0}[x_1, x_2, \ldots]$. The *key polynomial* $\kappa_\alpha := [\beta^0]\mathfrak{L}_\alpha$. The representation theoretic, geometric and combinatorial perspectives of Schubert polynomials and key polynomials are well-studied [5, 25, 1].

Define $V_n := \mathbb{Q}\text{-span}\{\mathfrak{S}_w : w \in S_n\}$ and $V := \mathbb{Q}\text{-span}\{\mathfrak{S}_w : w \in S_+\} = \bigcup_{n\geqslant 1} V_n$. In fact, $V = \mathbb{Q}[x_1, x_2, \ldots]$. By the increasing sequence $V_1 \subset V_2 \subset \cdots \subset V$, V has the structure of a filtered algebra.

In this paper, we are interested in the top-degree components of $\mathfrak{G}_w$ and $\mathfrak{L}_\alpha$. For a polynomial $f \in \mathbb{Q}[x_1, x_2, \ldots][\beta]$, let $\widehat{f} = [\beta^d](f)$ where d is the largest such that $[\beta^d](f) \neq 0$. The *Castelnuovo–Mumford polynomial* of $w \in S_+$ is defined as $\widehat{\mathfrak{G}}_w$. The *top Lascoux polynomial* of $\alpha \in C_+$ is defined as $\widehat{\mathfrak{L}}_\alpha$. In appendix §8, we list some Grothendieck polynomials and Lascoux polynomials. Pechenik, Speyer and Weigandt [18] first study $\widehat{\mathfrak{G}}_w$. To the best of the authors knowledge, $\widehat{\mathfrak{L}}_\alpha$ has not been studied previously.

Now consider the *tail lexicographic order* on monomials in the x-variables. We say a monomial x^α is larger than x^γ if there exists k such that $\alpha_k > \gamma_k$ and $\alpha_j = \gamma_j$ for all $j > k$. The *leading monomial* of $f \in \mathbb{Q}[x_1, x_2, \ldots]$ is the largest monomial in f. Among the four homogeneous polynomials above, three of them have combinatorial rules for their leading terms:

(1) [1] The leading monomial of $\mathfrak{S}_w$ with $w \in S_n$ is $x^{\mathsf{invcode}(w)}$, where
$$\mathsf{invcode}(w)_i = |\{j : (i, j) \in \mathsf{Inv}(w)\}|.$$
(2) [14] The leading monomial of κ_α is x^α.
(3) [18] The leading monomial of $\widehat{\mathfrak{G}}_w$ is $x^{\mathsf{rajcode}(w)}$ defined as follows.

Definition 2.1. [18] *Let* $\mathrm{LIS}^w(q)$ *be the length of the longest increasing subsequence of* $w \in S_n$ *that starts with* q*. The* $\mathsf{rajcode}(w)$ *for* $w \in S_n$ *is a weak composition* *where* $\mathsf{rajcode}(w)_r := n + 1 - r - \mathrm{LIS}^w(w(r))$ *for* $r \in [n]$ *and* 0 *if* $r > n$*. Then* $\mathsf{raj}(w) := |\mathsf{rajcode}(w)|$.

Example 2.2. Consider $w = 3721564 \in S_7$. We have $\mathrm{LIS}^w(2) = 3$, so $\mathsf{rajcode}(w)_3 = 7 + 1 - 3 - 3 = 2$. All together, we get $\mathsf{rajcode}(w) = (4, 5, 2, 1, 1, 1)$ and $\mathsf{raj}(w) = 14$.

We will define $\mathsf{rajcode}(\cdot)$ on C_+ and show the leading monomial of $\mathfrak{L}_\alpha$ is $x^{\mathsf{rajcode}(\alpha)}$ in §4.

A connection between $\mathfrak{G}_w$ and $\mathfrak{L}_\alpha$ is established by Shimozono and Yu [24]. To describe this connection, we need the following notion.

Definition 2.3. *Let* $f, f_1, f_2, \dots$ *be polynomials in* $\mathbb{Z}_{\geqslant 0}[x_1, x_2, \dots]$*. We say* f expands positively *into* $\{f_1, f_2, \dots\}$ *if there exist* $c_1, c_2, \dots \in \mathbb{Z}_{\geqslant 0}$ *such that* $f = \sum_i c_i f_i$*.*

Now assume $f, f_1, f_2, \dots$ *are polynomials in* $\mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$*. We say* f expands positively *into* $\{f_1, f_2, \dots\}$ *if there exist* $g_1, g_2, \dots \in \mathbb{Z}_{\geqslant 0}[\beta]$ *such that* $f = \sum_i g_i f_i$*.*

Theorem 2.4 ([24]). *For* $w \in S_+$*,* $\mathfrak{G}_w$ *expands positively into* $\{\mathfrak{L}_\alpha : \alpha \in C_+\}$*.*

This result implies $\widehat{\mathfrak{G}}_w$ also expands positively into $\widehat{\mathfrak{L}}_\alpha$ by the following lemma whose proof is sufficiently elementary.

Lemma 2.5. *Let* $f, f_1, f_2, \dots$ *in* $\mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$*. If* f *expands positively into* $\{f_1, f_2, \dots\}$*, then* $\widehat{f}$ *expands positively into* $\widehat{f}_1, \widehat{f}_2, \dots$*.*

Corollary 2.6. *For* $w \in S_+$*,* $\widehat{\mathfrak{G}}_w$ *expands positively into* $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_+\}$*.*

Define $\widehat{V}_n := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{G}}_w : w \in S_n\}$ and $\widehat{V} := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{G}}_w : w \in S_+\} = \bigcup_{n \geqslant 1} \widehat{V}_n$. By work of Lascoux, Schützenberger [13] and Brion [2], the product $\mathfrak{G}_u \mathfrak{G}_v$ with $u \in S_m$ and $v \in S_n$ expands positively into $\mathfrak{G}_w$ with $w \in S_{m+n}$. By Lemma 2.5, $\widehat{\mathfrak{G}}_u \widehat{\mathfrak{G}}_v$ with $u \in S_m$ and $v \in S_n$ expands positively into $\widehat{\mathfrak{G}}_w$ with $w \in S_{m+n}$. Finally, we conclude the following.

Proposition 2.7. *The space* $\widehat{V}$ *is a filtered algebra with respect to the filtration* $\widehat{V}_1 \subset \widehat{V}_2 \subset \cdots \subset \widehat{V}$*.*

2.2. Diagrams. A *diagram* is a finite subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. We represent a diagram by putting a cell at row r and column c for each (r, c) in the diagram. The leftmost column (resp. topmost row) is called column 1 (resp. row 1). The *weight* of a diagram D, denoted as $\mathsf{wt}(D)$, is a weak composition whose i^{th} entry is the number of boxes in its row i. We recall two classical families of diagrams.

Each weak composition α is associated with a diagram called the *key diagram*, denoted as $D(\alpha)$. It is the unique left-justified diagram with weight α. One important key diagram we will use later is $\mathsf{Stair}_n := D((n-1, n-2, \cdots, 1))$.

Example 2.8. The following are two examples of key diagrams. For clarity, we put an "i" on the left of row i and put a small dot in each cell.

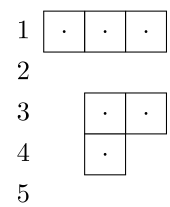
$$D(0,2,1) = \begin{array}{c|c|c|} 1 & \multicolumn{2}{c}{} \\ \cline{2-3} 2 & \cdot & \cdot \\ \cline{2-3} 3 & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} , \qquad \mathsf{Stair}_4 = \begin{array}{c|c|c|c|} \cline{2-4} 1 & \cdot & \cdot & \cdot \\ \cline{2-4} 2 & \cdot & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c}{} \\ \cline{2-3} 3 & \multicolumn{1}{c|}{\cdot} & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}$$

Each permutation w is associated with the *Rothe diagram* $RD(w) := \{(r, w(r')) : (r, r') \in \mathsf{Inv}(w)\}$.

Example 2.9. Let $w = 41532 \in S_5$. Then

$$\mathsf{Inv}(w) = \{(1,2), (1,4), (1,5), (3,4), (3,5), (4,5)\}.$$

The Rothe diagram is depicted as follows.

$RD(w) =$ 

2.3. K-Kohnert diagrams. We recall a combinatorial formula for Lascoux polynomials. To simplify our description, we introduce the following definition.

Definition 2.10. *A labeled diagram is a diagram where each cell can be labeled by a symbol. The underlying diagram of a labeled diagram is the diagram obtained by ignoring all labels. The weight of a labeled diagram D, denoted as $\mathsf{wt}(D)$, is just the weight of its underlying diagram.*

Then a *ghost diagram* is a labeled diagram where cells can be labeled by X. We call cells labeled by X as "*ghosts*". For a ghost diagram D, its *excess*, denoted as $\mathsf{ex}(D)$, is the number of ghosts in D. Next, we define a move on ghost diagrams.

Definition 2.11 ([21]). *A K-Kohnert move is defined on a ghost diagram D.*

We pick a cell (r, c) and move it up, subject to the following requirements.

- *The cell (r, c) must be the rightmost cell in row r.*
- *The cell (r, c) is not a ghost.*
- *The cell (r, c) is moved to the lowest empty spot above it.*
- *The cell (r, c) may jump over other cells but cannot jump over any ghosts.*

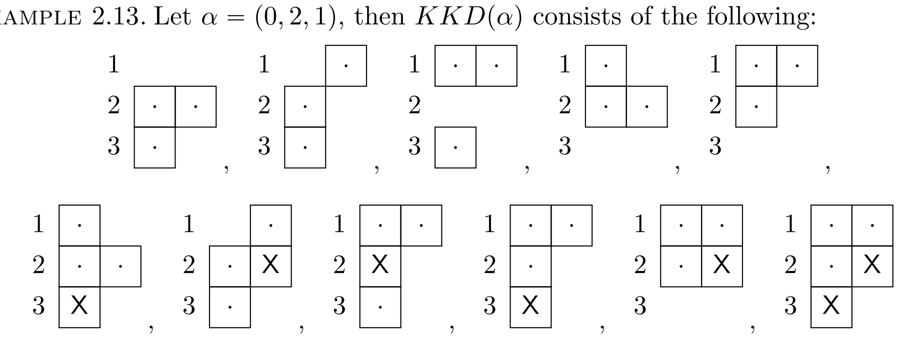
After the move, we may or may not leave a ghost at (r, c). When we leave a ghost, we refer this move as a ghost move.

For a weak composition α, a ghost diagram is called a *K-Kohnert diagram* of α if it can be obtained from $D(\alpha)$ by K-Kohnert moves. Let $\mathsf{KKD}(\alpha)$ be the set of all K-Kohnert diagrams of α. As proved in [17], K-Kohnert diagrams give a formula for Lascoux polynomials. This rule was first conjectured by Ross and Yong [21]. Notice that our convention is different from [17]: row 1 is the top most row in this paper while it is the bottom most row in [17].

Theorem 2.12 ([17]). *Let α be a weak composition. Then we have*

$$\mathfrak{L}_\alpha = \sum_{D \in \mathsf{KKD}(\alpha)} x^{\mathsf{wt}(D)} \beta^{\mathsf{ex}(D)}.$$

Example 2.13. Let $\alpha = (0, 2, 1)$, then $KKD(\alpha)$ consists of the following:



By the rule above, we have

$$\begin{aligned}\mathfrak{L}_\alpha &= x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2 \\ &\quad + \beta(x_1 x_2^2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2^2) + \beta^2 x_1^2 x_2^2 x_3.\end{aligned}$$

3. Snow diagrams

We associate each diagram with a labeled diagram called the *snow diagram* which allows us to define two statistics on diagrams. For each diagram D, we describe the following algorithm that outputs $\mathsf{snow}(D)$. Cells in $\mathsf{snow}(D)$ can be labeled by $\bullet$ or $\ast$.

- Iterate through rows of D from bottom to top.
- In each row r of D, find the rightmost cell (r, c) with no $\bullet$ in column c. If such an (r, c) exists, label it by $\bullet$ and put a cell labeled by $\ast$ in (r', c) for $r' \in [r-1]$ and $(r', c) \notin D$.

We call cells labeled by $\bullet$ *dark clouds* and cells labeled by $\ast$ *snowflakes*.

Example 3.1. The following is a diagram together with its snow diagram.

$$D = \begin{array}{c|c|c|c|} \cline{4-4} 1 & \multicolumn{1}{c}{} & & \cdot \\ \cline{2-4} 2 & \cdot & \cdot & \\ \cline{2-4} 3 & \multicolumn{1}{c}{} & & \cdot \\ \cline{4-4} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-3} 5 & \cdot & \cdot & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}, \quad \mathsf{snow}(D) = \begin{array}{c|c|c|c|} \cline{2-4} 1 & \ast & \ast & \cdot \\ \cline{2-4} 2 & \bullet & \cdot & \ast \\ \cline{2-4} 3 & \multicolumn{1}{c|}{} & \ast & \bullet \\ \cline{3-4} 4 & \multicolumn{1}{c|}{} & \ast & \multicolumn{1}{c}{} \\ \cline{2-3} 5 & \cdot & \bullet & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}.$$

The positions of dark clouds will be important, so we make the following definition.

Definition 3.2. *The* dark cloud diagram *of a diagram D, $\mathsf{dark}(D)$, is the set of cells (r, c) that are dark clouds in $\mathsf{snow}(D)$.*

Example 3.3. In Example 3.1, $\mathsf{dark}(D) = \{(2, 1), (3, 3), (5, 2)\}$.

A diagram is a *non-attacking rook diagram* if it has at most one cell in each row or column. Let Rook_+ be the family of all non-attacking rook diagrams.

Remark 3.4. We make the following observations about $\mathsf{dark}(D)$.

- By construction, $\mathsf{dark}(D) \in \mathsf{Rook}_+$.
- Take $(r, c) \in D$. If there are no $r' > r$ with $(r', c) \in \mathsf{dark}(D)$ and there are no $c' > c$ with $(r, c') \in \mathsf{dark}(D)$, then $(r, c) \in \mathsf{dark}(D)$.

Finally, we associate two statistics to each diagram via its snow diagram.

Definition 3.5. *Let D be a diagram. The* rajcode *of D, $\mathsf{rajcode}(D)$, is the weak composition $\mathsf{wt}(\mathsf{snow}(D))$. Let $\mathsf{raj}(D)$ denote $|\mathsf{rajcode}(D)|$, the total number of cells in $\mathsf{snow}(D)$.*

Example 3.6. Continuing with Example 3.1, we have $\mathsf{rajcode}(D) = (3, 3, 2, 1, 2)$ and $\mathsf{raj}(D) = 11$.

Remark 3.7. Recall that Pechenik, Speyer and Weigandt [18] define the statistics $\mathsf{rajcode}(\cdot)$ and $\mathsf{raj}(\cdot)$ on permutations using increasing subsequences. We show that our $\mathsf{rajcode}$ and raj on Rothe diagrams agree with their definitions in Theorem 5.6. Therefore, our construction on Rothe diagrams is a diagrammatic way to compute the leading monomial and degree of $\widehat{\mathfrak{G}}_w$. In addition, we notice that positions of dark clouds in $\mathsf{snow}(RD(w))$ are connected to the Schensted insertion and Viennot's geometric construction. These connections are explored in §5.

4. Proof of Theorem 1.2

To prove Theorem 1.2, we study top Lascoux polynomials via snow diagrams of key diagrams. With a slight abuse of notation, we define $\mathsf{rajcode}(\alpha) := \mathsf{rajcode}(D(\alpha))$, $\mathsf{raj}(\alpha) := \mathsf{raj}(D(\alpha))$ and $\mathsf{dark}(\alpha) = \mathsf{dark}(D(\alpha))$ for $\alpha \in C_+$. We start by introducing some definitions.

Definition 4.1. *A weak composition α is called* snowy *if its positive entries are all distinct.*

Our main goal in this section is to establish Theorem 1.2:

Theorem 1.2. *Let α and γ be two weak compositions.*

(a) *The polynomial $\widehat{\mathfrak{L}}_\alpha$ has leading monomial $x^{\mathsf{rajcode}(\alpha)}$.*
(b) *We have $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{L}}_\gamma$ if and only if $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$.*
(c) *We say α is snowy if its positive entries are distinct. If α is snowy, then $x^{\mathsf{rajcode}(\alpha)}$ has coefficient 1 in $\widehat{\mathfrak{L}}_\alpha$. Moreover, there exists exactly one snowy weak composition γ' such that $\mathsf{rajcode}(\gamma) = \mathsf{rajcode}(\gamma')$.*

This task is broken into four major lemmas established in the following four subsections. In Subsection 4.1, we use K-Kohnert diagrams to establish the first major lemma:

Lemma 4.2. *The polynomial $\mathfrak{L}_\alpha$ has the term $x^{\mathsf{rajcode}(\alpha)}\beta^{\mathsf{raj}(\alpha)-|\alpha|}$.*

Lemma 4.2 proves $\widehat{\mathfrak{L}}_\alpha$ has degree at least $\mathsf{raj}(\alpha)$. To show $\widehat{\mathfrak{L}}_\alpha$ indeed has degree $\mathsf{raj}(\alpha)$, we need the following equivalence relation on weak compositions.

Definition 4.3. *Let α and γ be two weak compositions. We say α is* rajcode equivalent *to γ, denoted as $\alpha \sim \gamma$, if $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$.*

Example 4.4. Let $\alpha = (2, 0, 4, 3, 1)$ and $\gamma = (3, 1, 4, 3, 1)$. Then we have:

$$D(\alpha) = \begin{array}{c|c|c|c|c|} 1 & \cdot & \cdot & \multicolumn{2}{c}{} \\ 2 & \multicolumn{4}{c}{} \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \multicolumn{1}{c}{} \\ 5 & \cdot & \multicolumn{3}{c}{} \end{array}\ , \qquad \mathsf{snow}(D(\alpha)) = \begin{array}{c|c|c|c|c|} 1 & \cdot & \bullet & \ast & \ast \\ 2 & \ast & & \ast & \ast \\ 3 & \cdot & \cdot & \cdot & \bullet \\ 4 & \cdot & \cdot & \bullet & \multicolumn{1}{c}{} \\ 5 & \bullet & \multicolumn{3}{c}{} \end{array}\ ,$$

$$D(\gamma) = \begin{array}{c|c|c|c|c|} 1 & \cdot & \cdot & \cdot & \multicolumn{1}{c}{} \\ 2 & \cdot & \multicolumn{3}{c}{} \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \multicolumn{1}{c}{} \\ 5 & \cdot & \multicolumn{3}{c}{} \end{array}\ , \qquad \mathsf{snow}(D(\gamma)) = \begin{array}{c|c|c|c|c|} 1 & \cdot & \bullet & \cdot & \ast \\ 2 & \cdot & & \ast & \ast \\ 3 & \cdot & \cdot & \cdot & \bullet \\ 4 & \cdot & \cdot & \bullet & \multicolumn{1}{c}{} \\ 5 & \bullet & \multicolumn{3}{c}{} \end{array}\ .$$

Be aware that the cell $(2, 2)$ is not in $\mathsf{snow}(D(\alpha))$ or $\mathsf{snow}(D(\gamma))$. Observe that $\mathsf{rajcode}(\alpha) = (4, 3, 4, 3, 1) = \mathsf{rajcode}(\gamma)$, so $\alpha \sim \gamma$.

In Subsection 4.2, we study this equivalence relation. We show that snowy weak compositions form a complete set of representatives:

Lemma 4.5. *For each equivalence class of $\sim$, there is a unique α such that α is snowy. Moreover, if $\gamma \sim \alpha$ and α is snowy, then $\gamma_r \geqslant \alpha_r$ for all r. In other words, a snowy weak composition is the unique entry-wise minimum in each equivalence class.*

In Subsection 4.3, we focus on $\widehat{\mathfrak{L}}_\alpha$ for snowy α and give a recursive description of $\widehat{\mathfrak{L}}_\alpha$, which leads to the third major lemma.

Lemma 4.6. *If α is snowy, then $x^{\mathsf{rajcode}(\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$ with coefficient 1.*

Finally, we devote the Subsection 4.4 to proving the last major lemma:

Lemma 4.7. *If $\alpha \sim \gamma$, then $\widehat{\mathfrak{L}}_\alpha = c\widehat{\mathfrak{L}}_\gamma$ for some $c \neq 0$.*

Once we have these four major lemmas, we can easily check Theorem 1.2.

Proof. First, statement (c) follows from Lemma 4.5 and Lemma 4.6.

Given a weak composition α. Let β be the unique snowy weak composition such that $\alpha \sim \beta$. Statement (a) follows from Lemma 4.6 and Lemma 4.7.

For statement (b), the backward direction is just Lemma 4.7. For the forward direction, if $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{L}}_\gamma$, then they have the same leading monomial. By statement (a), we have $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$. $\square$

4.1. Proof of Lemma 4.2. We show the monomial $x^{\mathsf{rajcode}(\alpha)}\beta^{\mathsf{raj}(\alpha)-|\alpha|}$ exists in $\mathfrak{L}_\alpha$. We give an algorithm whose output is a K-Kohnert diagram for α, which has the same underlying diagram as $\mathsf{snow}(D(\alpha))$. First, observe that $\mathsf{snow}(D(\alpha))$ contains no dark clouds if and only if α contains only zero entries. In this case, $\widehat{\mathfrak{L}}_\alpha = 1$ and $\mathsf{rajcode}(\alpha)$ only has zero entries. Our claim is immediate. In the rest of this subsection, we assume α is a weak composition with at least one positive entry, and thus $\mathsf{snow}(D(\alpha))$ has at least one dark cloud. To describe the algorithm, we introduce two useful moves on ghost diagrams.

Definition 4.8. *Let D be a ghost diagram. Let (r, c) be a non-ghost cell in D and let (r', c) be the highest empty space in column c. If $r' < r$, let $UP_{(r,c)}(D)$ be the diagram we get after moving (r, c) to (r', c). Let $UP^G_{(r,c)}(D)$ be the diagram we get after moving (r, c) to (r', c) and putting a ghost on (r, c) and all empty spaces between (r, c) and (r', c). If $r' > r$, define $UP^G_{(r,c)}(D) = UP_{(r,c)}(D) = D$.*

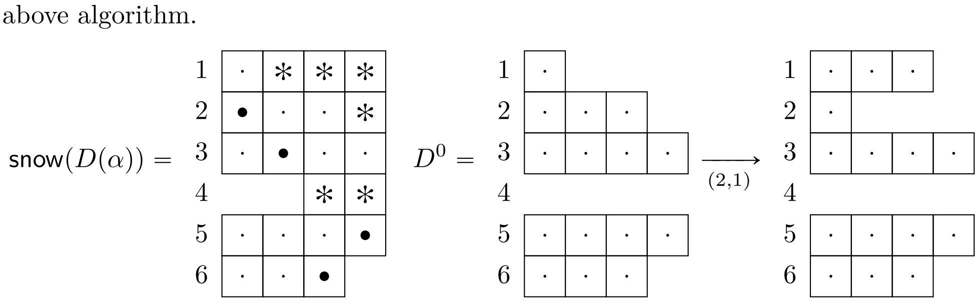
Remark 4.9. Assume $UP_{(r,c)}$ or $UP^G_{(r,c)}$ moves a cell to (r', c). Then this move can be achieved by a sequence of K-Kohnert moves if both of the following conditions hold for each $r' < j \leqslant r$:

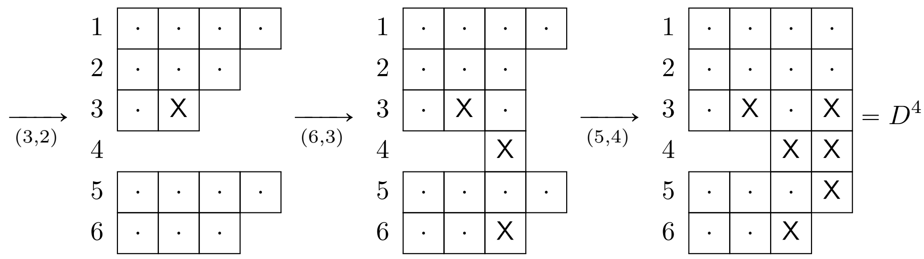
- If $(j, c) \notin D$, then D has no cell to the right of column c in row j.
- If $(j, c) \in D$, then it is not a ghost cell.

Now we can describe the algorithm. Let $D^0 = D(\alpha)$. Recall by Remark 3.4, there is at most one dark cloud in each column of $\mathsf{snow}(D(\alpha))$. We can label all the dark clouds as $(r_1, c_1), \ldots, (r_m, c_m)$ where $c_1 < c_2 \cdots < c_m$ for some $m \geqslant 1$. We iterate i from 1 to m. At iteration i, compute

$$D^i = UP^G_{(r_i,c_i)} \circ UP_{(r_i,c_i+1)} \cdots \circ UP_{(r_i,\alpha_{r_i})}(D^{i-1}). \qquad (1)$$

Example 4.10. Consider $\alpha = (1, 3, 4, 0, 4, 3)$, we compute its snow diagram and we have the dark clouds at $(2, 1), (3, 2), (6, 3), (5, 4)$. We compute D^4 according to the above algorithm.





We observe that in the previous example, D^4 has the same underlying diagram as $\mathsf{snow}(D(\alpha))$. This is true in general.

Lemma 4.11. *The labeled diagram D^m defined by (1) has the same underlying diagram as $\mathsf{snow}(D(\alpha))$.*

Proof. For a number c, we compare the column c of $\mathsf{snow}(D(\alpha))$ and D^m. If column c of $\mathsf{snow}(D(\alpha))$ has no dark cloud, then it is the same as column c of $D(\alpha)$. In this case, the algorithm will not move any cells in column c. Thus, D^m and $D(\alpha)$ also agree in column c.

Now suppose $\mathsf{snow}(D(\alpha))$ has a dark cloud in column c, say at row r. In the underlying diagram of $\mathsf{snow}(D(\alpha))$, column c is obtained from column c of $D(\alpha)$ by filling all empty spaces above row r. On the other hand, consider what the algorithm does on column c. It first might move cells above row r and then it fills all empty spaces weakly above row r. Thus, column c in D^m is the same as column c of $\mathsf{snow}(D(\alpha))$ after ignoring the labels. □

Next, we want to show D^m produced by the algorithm is in $\mathsf{KKD}(\alpha)$. We just need to check each $UP_{(r,j)}$ and $UP^G_{(r,c)}$ in each iteration is a sequence of K-Kohnert moves. To that end, we first make the following observation about the diagram D^i.

Lemma 4.12. *Let $c_0 = 0$. In D^i, if a cell is strictly to the right of column c_i, then there is a cell immediately on its left. In other words, the diagram D^i is left-justified if we ignore the first c_i columns.*

Proof. Prove by induction on i. The lemma holds for D^0, which is left-justified.

Assume D^{i-1} is left-justified if we ignore the first c_{i-1} columns, for some $i \geqslant 1$. Consider an arbitrary cell (r,c) in D^i with $c > c_i$. We show $(r, c-1)$ is in D^i by considering two possibilities.

- The cell (r,c) is not in D^{i-1}. Then during iteration i, a cell is moved to (r,c), which is the highest blank in column c of D^{i-1}. By our inductive hypothesis and $c-1 > c_{i-1}$, the highest blank in column $c-1$ of D^{i-1} is weakly lower than row r. Thus, $(r, c-1)$ is in D^i.
- Otherwise, (r,c) is in D^{i-1}. By our inductive hypothesis, $(r, c-1)$ is in D^{i-1}. If $r \neq r_i$, then we know that no cell from row r is moved during iteration i. Thus, $(r, c-1)$ is still in D^i. If $r = r_i$, then there are no empty spaces above (r,c) in D^{i-1}. By our inductive hypothesis, there is no empty spaces above $(r, c-1)$, so $(r, c-1)$ is still in D^i. □

The above lemma shows that the diagram D^i is left-justified if we ignore the first c_i columns. We will use this property to show that D^m is in $\mathsf{KKD}(\alpha)$.

Proposition 4.13. *The above algorithm can be achieved by K-Kohnert moves, so $D^m \in \mathsf{KKD}(\alpha)$.*

Proof. We focus on one iteration of the algorithm, say iteration i. We check the operators in (1) can be achieved by K-Kohnert moves. We ignore all cells to the left of the column c_i in D^{i-1}. By the previous Lemma, this part of the diagram is left-justified. The highest empty spaces in columns $c_i, \cdots, \alpha_{r_i}$ are going weakly up from left to right. Moreover, the condition in Remark 4.9 holds for all $(r_i, c_i), \cdots, (r_i, \alpha_{r_i})$.

Now $UP_{(r_i,\alpha_{r_i})}$ can be achieved by K-Kohnert moves. After that, the conditions in Remark 4.9 hold at each step for $(r_i, \alpha_{r_i} - 1), \ldots, (r_i, c_i)$. Following this logic, this iteration can be achieved by K-Kohnert moves. $\square$

Using Theorem 2.12:

Lemma 4.2. *The polynomial $\mathfrak{L}_\alpha$ has the term $x^{\mathsf{rajcode}(\alpha)}\beta^{\mathsf{raj}(\alpha)-|\alpha|}$.*

4.2. Proof of Lemma 4.5. First, notice that we can recover the underlying diagram of $\mathsf{snow}(D(\alpha))$ from $\mathsf{dark}(\alpha)$.

Lemma 4.14. *Let α be a weak composition. The underlying diagram of $\mathsf{snow}(D(\alpha))$ is:*

$$\bigcup_{(r,c)\in\mathsf{dark}(\alpha)} ([r] \times \{c\}) \cup (\{r\} \times [c]). \tag{2}$$

Proof. First, we show that the elements of the set (2) are cells in $\mathsf{snow}(D(\alpha))$. Take $(r, c) \in \mathsf{dark}(\alpha)$. We know $(r, c) \in D(\alpha)$. Since $D(\alpha)$ is left-justified, $\{r\} \times [c] \subseteq D(\alpha)$. Thus, these cells are in $\mathsf{snow}(D(\alpha))$. By the construction of $\mathsf{snow}(D(\alpha))$, the cells in $[r] \times \{c\}$ are also in $\mathsf{snow}(D(\alpha))$.

Now suppose there is a cell (r, c) in $\mathsf{snow}(D(\alpha))$ that is not in the set (2). Then there is no $r' > r$ with $(r', c) \in \mathsf{dark}(D)$, which implies (r, c) is not a snowflake in $\mathsf{snow}(D(\alpha))$. Thus, $(r, c) \in D(\alpha)$. Also, there is no $c' > c$ with $(r, c') \in \mathsf{dark}(D)$. By Remark 3.4, $(r, c) \in \mathsf{dark}(D)$. Thus, (r, c) is in the set (2), which is a contradiction. $\square$

Furthermore, we can recover $\mathsf{dark}(\alpha)$ from $\mathsf{rajcode}(\alpha)$.

Lemma 4.15. *Let α, γ be weak compositions. If $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$, then $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$.*

Proof. We prove the two diagrams $\mathsf{dark}(\alpha)$ and $\mathsf{dark}(\gamma)$ agree on each row r, by a reverse induction on r. The base case is immediate. Suppose r is large enough such that $\alpha_i = \gamma_i = 0$ if $i > r$. Then $\mathsf{dark}(\alpha)$ and $\mathsf{dark}(\gamma)$ clearly agree on row r and underneath.

Next, we show that the value $\mathsf{rajcode}(\alpha)_r$ and cells in $\mathsf{dark}(\alpha)$ under row r determines whether $\mathsf{dark}(\alpha)$ has a cell on row r. Moreover, if such a cell exists, its column index is also determined.

Let $r \geqslant 1$. Define

$$B_r := \{c : \text{There are no dark clouds under } (r, c) \text{ in } \mathsf{snow}(D(\alpha)\}.$$

The complement of B_r is $\overline{B_r} := \mathbb{Z}_{>0} - B_r = \{c : (r', c) \in \mathsf{dark}(\alpha) \text{ for some } r' > r\}$. For $c \in \overline{B_r}$, (r, c) of $\mathsf{snow}(D(\alpha))$ is a snowflake or an unlabeled cell. If there is no dark cloud on row r of $\mathsf{snow}(D(\alpha))$, $\mathsf{rajcode}(\alpha)_r = |\overline{B_r}|$. Otherwise, we assume the dark cloud is at (r, c) for some $c \in B_r$. Then row r of $\mathsf{snow}(D(\alpha))$ has cells on (r, c') for $c' \in \overline{B_r}$ or $c' \leqslant c$. Suppose c is the i^{th} smallest number in B_r. We have $\mathsf{rajcode}(\alpha)_r = i + |\overline{B_r}|$.

Consequently, $\mathsf{rajcode}(\alpha)_r$ and $\mathsf{dark}(\alpha)$ under row r uniquely determines row r of $\mathsf{dark}(\alpha)$. If we assume $\mathsf{dark}(\alpha)$ and $\mathsf{dark}(\gamma)$ agree underneath row r as our inductive hypothesis, then they also agree on row r since $\mathsf{rajcode}(\alpha)_r = \mathsf{rajcode}(\gamma)_r$. The induction is finished. $\square$

Now we have two equivalent ways of describing rajcode equivalence.

Proposition 4.16. *Let α and γ be two weak compositions. The following are equivalent:*

(1) *$\alpha \sim \gamma$;*
(2) $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$.
(3) *The underlying diagrams of* $\mathsf{snow}(D(\alpha))$ *and* $\mathsf{snow}(D(\gamma))$ *are the same;*

Proof. By Lemma 4.15, (1) implies (2). By Lemma 4.14, (2) implies (3). Clearly, (3) implies (1). □

Our next goal is to find representatives of rajcode equivalence classes. At the end of this subsection, we will see snowy weak compositions form a complete set of representatives. To understand snowy weak compositions, we start with the following observation.

Remark 4.17. For a weak composition α, the following are equivalent:

- α is snowy.
- The rightmost cell in each row of $D(\alpha)$ are in different columns.
- The rightmost cell in each row of $D(\alpha)$ is a dark cloud in $\mathsf{snow}(D(\alpha))$.

One advantage of working with snowy weak compositions is that we can tell their $\mathsf{rajcode}(\cdot)$ and $\mathsf{raj}(\cdot)$ easily:

Lemma 4.18. *Let α be a snowy weak composition. Then the following statements hold.*

(1) $\mathsf{dark}(\alpha) = \{(r, \alpha_r) : \alpha_r > 0\}$,
(2) $\mathsf{rajcode}(\alpha)_r = \alpha_r + |\{r' > r : \alpha_r < \alpha'_r\}|$, *and*
(3) $\mathsf{raj}(\alpha) = \sum_r(\alpha_r + |\{(r, r') : \alpha_r < \alpha'_r, r < r'\}|) = |\alpha| + |\{(r, r') : r < r', \alpha_r < \alpha_{r'}\}|$.

Proof. (1) follows from Remark 4.17. (2) follows from (1) and Lemma 4.14, and (3) immediately follows from (2). □

As a consequence, we have the following rule which tells us how $\mathsf{rajcode}(s_i\alpha)$ differs from $\mathsf{rajcode}(\alpha)$ when α is snowy.

Corollary 4.19. *Let α be a snowy weak composition and consider i with $\alpha_i > \alpha_{i+1}$. Then* $\mathsf{rajcode}(s_i\alpha) = s_i\mathsf{rajcode}(\alpha) + e_i$, *where e_i is the weak composition with 1 on its i^{th} entry and 0 elsewhere.*

The second advantage of working with snowy weak compositions is that they are in bijection with Rook_+.

Lemma 4.20. *The map* $\mathsf{dark}(\cdot)$ *is a bijection from $\{\alpha \in C_+ : \alpha$ is snowy$\}$ to* Rook_+. *Its inverse* $\mathsf{dark}^{-1}(\cdot)$ *is given by* $\mathsf{dark}^{-1}(R) = \alpha$ *where*

$$\alpha_r = \begin{cases} 0 & \text{if row } r \text{ of } R \text{ is empty;} \\ c & \text{if } (r, c) \in R. \end{cases}$$

Proof. Follows from Remark 4.17. □

We are ready to show that they are representatives of all equivalence classes.

Lemma 4.5. *For each equivalence class of $\sim$, there is a unique α such that α is snowy. Moreover, if $\gamma \sim \alpha$ and α is snowy, then $\gamma_r \geqslant \alpha_r$ for all r. In other words, a snowy weak composition is the unique entry-wise minimum in each equivalence class.*

Proof. Let γ be an arbitrary weak composition. First, we construct a snowy α such that $\alpha \sim \gamma$. We know $\mathsf{dark}(\gamma) \in \mathsf{Rook}_+$. We send it to a snowy α using the map in Lemma 4.20. Then $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$. By Proposition 4.16, $\alpha \sim \gamma$.

Next, take a positive integer r. If $\alpha_r = 0$, then $\gamma_r \geqslant \alpha_r$ trivially. Otherwise, we know $(r, \alpha_r) \in \mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$. Thus, $\gamma_r \geqslant \alpha_r$.

Finally, we establish the uniqueness of this snowy α. Assume α' is a snowy weak composition such that $\alpha' \sim \gamma$. Then $\alpha'_r \geqslant \alpha_r$ and $\alpha_r \geqslant \alpha'_r$ for all $r \in \mathbb{Z}_{>0}$, so $\alpha = \alpha'$. □

A snowy weak composition has more snowflakes in its snow diagram than any others in its equivalence class; hence the name. Say $\alpha \sim \gamma$ and α is snowy while γ is not. By Lemma 4.5, $|\alpha| < |\gamma|$. On the other hand, the number of snowflakes in $\mathsf{snow}(D(\alpha))$ (resp. $\mathsf{snow}(D(\gamma))$) is $\mathsf{raj}(\alpha) - |\alpha|$ (resp. $\mathsf{raj}(\gamma) - |\gamma|$). Since $\mathsf{raj}(\alpha) = \mathsf{raj}(\gamma)$, $\mathsf{snow}(D(\alpha))$ has more snowflakes than $\mathsf{snow}(D(\gamma))$.

4.3. Proof of Lemma 4.6. By Lemma 4.2, $\widehat{\mathfrak{L}}_\alpha$ has degree at least $\mathsf{raj}(\alpha)$. Next, we can show the degree of $\widehat{\mathfrak{L}}_\alpha$ equals to $\mathsf{raj}(\alpha)$ when α is snowy.

Lemma 4.21. *Let α be a snowy weak composition. The β-degree of $\mathfrak{L}_\alpha$ is $\mathsf{raj}(\alpha) - |\alpha|$, so the degree of $\widehat{\mathfrak{L}}_\alpha$ is $\mathsf{raj}(\alpha)$.*

Proof. We prove the result by induction on

$$\ell(\alpha) := |\{(i,j) \mid \alpha_i < \alpha_j \text{ and } i < j\}|.$$

For the base case, if $\ell(\alpha) = 0$, then α is weakly decreasing. The polynomial $\mathfrak{L}_\alpha$ is an monomial with β-degree 0. Correspondingly, $\mathsf{raj}(\alpha) = |\alpha|$.

Now if $\ell(\alpha) > 0$, we can find i with $\alpha_i < \alpha_{i+1}$. By Corollary 4.19, $\mathsf{raj}(s_i\alpha) = \mathsf{raj}(\alpha) - 1$. Notice that $\ell(s_i\alpha) = \ell(\alpha) - 1$. By our inductive hypothesis, the β-degree of $\mathfrak{L}_{s_i\alpha}$ is $\mathsf{raj}(s_i\alpha) - |\alpha| = \mathsf{raj}(\alpha) - 1 - |\alpha|$. By the recursive definition of Lascoux polynomials,

$$\mathfrak{L}_\alpha = \pi_i(\mathfrak{L}_{s_i\alpha}) + \beta\pi_i(x_{i+1}\mathfrak{L}_{s_i\alpha}).$$

The β-degree in $\mathfrak{L}_\alpha$ is at most $\mathsf{raj}(\alpha) - |\alpha|$. Lemma 4.2 implies the β-degree of $\mathfrak{L}_\alpha$ is at least $\mathsf{raj}(\alpha) - |\alpha|$, so the inductive step is finished. □

Combine with Lemma 4.18, we have:

Corollary 4.22. *Let α be a snowy weak composition. The degree of $\widehat{\mathfrak{L}}_\alpha$ is $|\alpha| + |\{(r,r') : r < r', \alpha_r < \alpha_{r'}\}|$.*

Now we can describe $\widehat{\mathfrak{L}}_\alpha$ for snowy α recursively.

Lemma 4.23. *Let α be a snowy weak composition. Then*

$$\widehat{\mathfrak{L}}_\alpha = \begin{cases} x^\alpha & \text{if } \alpha_1 \geqslant \alpha_2 \geqslant \cdots \\ \pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases} \tag{3}$$

Proof. When α is weakly decreasing, our rule is immediate. Now assume $\alpha_i < \alpha_{i+1}$ for some $i \in \mathbb{Z}_{>0}$. By Corollary 4.19, $\mathsf{raj}(s_i\alpha) = \mathsf{raj}(\alpha) - 1$. We write $\mathfrak{L}_{s_i\alpha}$ as $g + \beta^{\mathsf{raj}(\alpha)-1-|\alpha|}\widehat{\mathfrak{L}}_{s_i\alpha}$ for some $g \in \mathbb{Z}[x_1, x_2, \cdots][\beta]$ with β-degree less than $\mathsf{raj}(\alpha) - 1 - |\alpha|$. Now we write $\mathfrak{L}_\alpha$ as

$$\begin{aligned} \mathfrak{L}_\alpha &= \pi_i(\mathfrak{L}_{s_i\alpha}) + \beta\pi_i(x_{i+1}\mathfrak{L}_{s_i\alpha}) \\ &= \pi_i(\mathfrak{L}_{s_i\alpha}) + \beta\pi_i(x_{i+1}g) + \beta^{\mathsf{raj}(\alpha)-|\alpha|}\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_i\alpha}) \end{aligned}$$

When we extract the coefficient of $\beta^{\mathsf{raj}(\alpha)-|\alpha|}$, the left-hand side is $\widehat{\mathfrak{L}}_\alpha$. On the right-hand side, the first two terms are ignored and we get $\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_i\alpha})$. □

Combining Lemma 4.2 and Lemma 4.21, we know $x^{\mathsf{rajcode}(\alpha)}$ appears in $\widehat{\mathfrak{L}}_\alpha$ when α is snowy. Next, we show this monomial is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$. We start with the following observation about the operator $f \mapsto \pi_i(x_{i+1}f)$.

Remark 4.24. Let γ be a monomial. We may describe the leading monomial of $\pi_i(x_{i+1}x^\gamma)$ and its coefficient as follows.

- If $\gamma_i > \gamma_{i+1}$, then $x_i x^{s_i\gamma}$ is the leading monomial with coefficient 1.
- If $\gamma_i = \gamma_{i+1}$, then $\pi_i(x_{i+1}x^\gamma) = 0$.
- If $\gamma_i < \gamma_{i+1}$, then $x_i x^\gamma$ is the leading monomial with coefficient -1.

We can understand how the operator $f \mapsto \pi_i(x_{i+1}f)$ changes the leading monomial of polynomial f satisfying certain conditions.

Lemma 4.25. *Take $f \in \mathbb{Z}[x_1, x_2, \cdots]$ with $f \neq 0$. Assume x^α is the leading monomial in f with coefficient $c \neq 0$. Pick an $i \in \mathbb{Z}_{>0}$ such that $\alpha_i > \alpha_{i+1}$. Furthermore, assume for any monomial in f, its power of x_i is at most α_i. Then $x_i x^{s_i\alpha}$ is the leading monomial in $\pi_i(x_{i+1}f)$ with coefficient c.*

Proof. In this proof, we use "$\geqslant$" to denote the monomial order. Let Γ be the set of weak compositions γ such that x^γ appears in f. Let c_γ be the coefficient of x^γ in f. We may write $f = \sum_{\gamma\in\Gamma} c_\gamma x^\gamma$. Then $\pi_i(x_{i+1}f) = \sum_{\gamma\in\Gamma} c_\gamma \pi_i(x_{i+1}x^\gamma)$. By the remark above, $x_i x^{s_i\alpha}$ appears in $c_\alpha \pi_i(x_{i+1}x^\alpha)$ as the leading monomial with coefficient $c_\alpha = c$. It is enough to show the following claim.

Claim: Take $\gamma \in \Gamma$ such that $\pi_i(x_{i+1}x^\gamma) \neq 0$ (i.e. $\gamma_i \neq \gamma_{i+1}$). Let $x^{\gamma'}$ be the leading monomial in $\pi_i(x_{i+1}x^\gamma)$. If $x^{\gamma'} \geqslant x_i x^{s_i\alpha}$, then $\gamma = \alpha$.

Proof: Assume $\alpha \neq \gamma$. Let k be the largest index such that the power of x_k differs in $x^{\gamma'}$ and $x_i x^{s_i\alpha}$. By $x^{\gamma'} \geqslant x_i x^{s_i\alpha}$, the power of x_k in $x^{\gamma'}$ is greater than the power of x_k in $x_i x^{s_i\alpha}$. We must have $k \leqslant i+1$. Otherwise, $x^\gamma > x^\alpha$, which contradicts x^α being the leading monomial in f.

Now we know γ', α and γ all agree after the $(i+1)^{th}$ entry. Then γ'_{i+1} is at least the power of x_{i+1} in $x_i x^{s_i\alpha}$, which is α_i. On the other hand, by $x^\gamma \leqslant x^\alpha$, $\gamma_{i+1} \leqslant \alpha_{i+1}$. Thus,

$$\gamma_{i+1} \leqslant \alpha_{i+1} < \alpha_i \leqslant \gamma'_{i+1}. \tag{4}$$

If $\gamma_i < \gamma_{i+1}$, Remark 4.24 implies $\gamma'_{i+1} = \gamma_{i+1}$, which is impossible. Thus, we must have $\gamma_i > \gamma_{i+1}$. By Remark 4.24 again, $\gamma'_{i+1} = \gamma_i$. By the assumptions in the statement of the lemma, $\gamma_i \leqslant \alpha_i$, so $\gamma'_{i+1} = \gamma_i = \alpha_i$.

Next, γ'_i is at least the power of x_i in $x_i x^{s_i\alpha}$, which is $\alpha_{i+1}+1$. Remark 4.24 implies $\gamma'_i = \gamma_{i+1}+1$. Thus, $\gamma_{i+1} \geqslant \alpha_{i+1}$. By (4), $\gamma_{i+1} = \alpha_{i+1}$.

Now we know $k < i$ and $\gamma_j = \alpha_j$ for $j = i$ or $i+1$. Thus, $\gamma_j = \alpha_j$ for all $j > k$, so $x^\gamma > x^\alpha$, which is a contradiction. ☐

Now we can establish our third major lemma.

Lemma 4.6. *If α is snowy, then $x^{\mathsf{rajcode}(\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$ with coefficient 1.*

Proof. We prove the result by induction on

$$\ell(\alpha) := |\{(i,j) \mid \alpha_i < \alpha_j \text{ and } i < j\}|.$$

If $\ell(\alpha) = 0$, then α is weakly decreasing, then $\mathfrak{L}_\alpha = x^\alpha = x^{\mathsf{rajcode}(\alpha)}$. Our claim is immediate.

Now if $\ell(\alpha) > 0$, we can find r with $\alpha_r < \alpha_{r+1}$. Pick the largest such r. For our inductive hypothesis, assume $x^{\mathsf{rajcode}(s_r\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_{s_r\alpha}$ with coefficient 1.

By the maximality of r, $\alpha_{r+1} \geqslant \alpha_{r+2} \geqslant \alpha_{r+3} \geqslant \cdots$. Thus, in any K-Kohnert diagram of $s_r\alpha$, there cannot be more than α_{r+1} cells in row r. In other words, for any monomial of $\widehat{\mathfrak{L}}_{s_r\alpha}$, the power of x_r is at most α_{r+1}. Lemma 4.25 implies that $x_r x^{s_r \mathsf{rajcode}(s_r\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$ with coefficient 1. Finally, by Corollary 4.19, $x_r x^{s_r \mathsf{rajcode}(s_r\alpha)} = x^{\mathsf{rajcode}(\alpha)}$. ☐

4.4. Proof of Lemma 4.7. We first derive two consequences of $\alpha \sim \gamma$. We start with the following definition.

Definition 4.26. *Let D be a diagram. Let $\overline{D} := \bigcup_{(r,c)\in D}[r] \times \{c\}$.*

In plain words, $\overline{D}$ is the diagram obtained by filling the empty spaces above each cell of D. Then $\overline{D(\alpha)}$ is completely determined by $\mathsf{dark}(\alpha)$:

Lemma 4.27. *Let α be a weak composition. Then $\overline{D(\alpha)} = \bigcup_{(r,c)\in\mathsf{dark}(\alpha)}[r] \times [c]$.*

Proof. We show each side is a subset of the other. Take $(r_1, c_1) \in D(\alpha)$. By Remark 3.4, there is $(r_2, c_2) \in \mathsf{dark}(\alpha)$ such that $r_2 \geqslant r_1$ and $c_2 \geqslant c_1$. Thus, $[r_1] \times \{c_1\} \subseteq [r_2] \times [c_2]$.

Take $(r_1, c_1) \in \mathsf{dark}(\alpha)$. Thus, for any $c \in [c_1]$, $(r_1, c) \in D(\alpha)$. Then $[r_1] \times \{c\} \subseteq \overline{D(\alpha)}$, so $[r_1] \times [c_1] \subseteq \overline{D(\alpha)}$. ☐

We have the following consequence of $\alpha \sim \gamma$.

Corollary 4.28. *If $\alpha \sim \gamma$, then $\overline{D(\alpha)} = \overline{D(\gamma)}$.*

Notice that the converse is not true. If $\alpha = (1, 2)$ and $\gamma = (0, 2)$, then $\overline{D(\alpha)} = [2] \times [2] = \overline{D(\gamma)}$. However, α and γ are not similar, since $\mathsf{dark}(\alpha) = \{(1, 1), (2, 2)\}$ and $\mathsf{dark}(\gamma) = \{(2, 2)\}$.

Another nice consequence of $\alpha \sim \gamma$ one might expect is $s_r\alpha \sim s_r\gamma$. Unfortunately, this is not always true. It is easy to check $(0, 1) \sim (1, 1)$ but $s_1(0, 1) = (1, 0)$ and $s_1(1, 1) = (1, 1)$ are not similar. However, it is true when α and r satisfy the following condition.

Lemma 4.29. *Let α be a weak composition and $r \in \mathbb{Z}_{>0}$. Assume there exists c such that $(r, c) \notin \mathsf{snow}(D(\alpha))$ but $(r + 1, c) \in \mathsf{snow}(D(\alpha))$. Then*

- (i) *$\alpha_{r+1} > \alpha_r$;*
- (ii) *The diagram $\mathsf{dark}(s_r\alpha)$ is obtained from $\mathsf{dark}(\alpha)$ by switching row r and row $r + 1$;*
- (iii) *For any γ with $\gamma \sim \alpha$, we must have $\gamma_{r+1} > \gamma_r$ and $s_r\alpha \sim s_r\gamma$.*

Proof. Since (r, c) is not in $\mathsf{snow}(D(\alpha))$, we can deduce two facts:

(1) There are no dark clouds under row r in column c, and
(2) $\alpha_r < c$.

By (1), the cell $(r + 1, c)$ in $\mathsf{snow}(D(\alpha))$ is not a dark cloud or a snowflake. Thus, it is unlabeled and $(r + 1, c) \in D(\alpha)$. By Remark 3.4, there must be a $c' > c$ such that $(r + 1, c')$ is a dark cloud in $\mathsf{snow}(D(\alpha))$. This implies $\alpha_{r+1} > c$. By (2), we have $\alpha_{r+1} > \alpha_r$, proving (i). Also by (2), the dark cloud in row r of $\mathsf{snow}(D(\alpha))$, if exists, is in the first $c - 1$ columns. Thus, $\mathsf{dark}(s_r\alpha)$ is obtained from $\mathsf{dark}(\alpha)$ by switching row r and row $r + 1$, proving (ii).

Now consider any $\gamma \sim \alpha$. By Proposition 4.16, $\mathsf{snow}(D(\gamma))$ and $\mathsf{snow}(D(\alpha))$ have the same underlying diagram. By (ii), $\mathsf{dark}(s_r\gamma)$ is obtained from $\mathsf{dark}(\gamma)$ by switching row r and row $r + 1$. Since $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$, we have $\mathsf{dark}(s_r\alpha) = \mathsf{dark}(s_r\gamma)$, so $s_r\alpha \sim s_r\gamma$. ☐

These two consequences of $\alpha \sim \gamma$ allow us to prove the last main Lemma.

Lemma 4.7. *If $\alpha \sim \gamma$, then $\widehat{\mathfrak{L}}_\alpha = c\widehat{\mathfrak{L}}_\gamma$ for some $c \neq 0$.*

Proof. By Lemma 4.5 it is enough to assume γ is snowy, and we proceed by induction on $\mathsf{raj}(\alpha)$. The base case is $\mathsf{raj}(\alpha) = 0$, which implies α only has 0s. Our claim is immediate.

Now assume $\mathsf{raj}(\alpha) > 0$. Consider the diagram $\overline{D(\alpha)}$. Clearly, the underlying diagram of any K-Kohnert diagram of α will be a subset of $\overline{D(\alpha)}$. In other words, any monomial in $\widehat{\mathfrak{L}}_\alpha$ must divide $x^{\mathsf{wt}(\overline{D(\alpha)})}$.

If the underlying diagram of $\mathsf{snow}(D(\alpha))$ is $\overline{D(\alpha)}$, then $x^{\mathsf{wt}(\overline{D(\alpha)})}$ is the only monomial in $\widehat{\mathfrak{L}}_\alpha$. On the other hand, Corollary 4.28 gives $\overline{D(\alpha)} = \overline{D(\gamma)}$. By the same argument, $x^{\mathsf{wt}(\overline{D(\alpha)})}$ is the only monomial in $\widehat{\mathfrak{L}}_\gamma$. Our claim holds.

Otherwise, we can find $(r, c) \in \overline{D(\alpha)}$ but not in $\mathsf{snow}(D(\alpha))$. Choose the (r, c) with the largest r. First, we know $(r, c) \notin D(\alpha)$, which implies $(r + 1, c) \in \overline{D(\alpha)}$. By the maximality of r, $(r + 1, c)$ is in $\mathsf{snow}(D(\alpha))$. We invoke Lemma 4.29 and conclude $\alpha_{r+1} > \alpha_r$, $\gamma_{r+1} > \gamma_r$ and $s_i\alpha \sim s_i\gamma$. Since γ is snowy, by Corollary 4.19, we know $\mathsf{raj}(s_r\gamma) = \mathsf{raj}(\gamma) - 1$, which implies $\mathsf{raj}(s_r\alpha) = \mathsf{raj}(\alpha) - 1$. By our inductive hypothesis, $\widehat{\mathfrak{L}}_{s_r\alpha} = c\widehat{\mathfrak{L}}_{s_r\gamma}$ for some $c \neq 0$.

We may write $\mathfrak{L}_{s_r\alpha}$ as $\beta^{\mathsf{raj}(s_r\alpha)-|\alpha|}\widehat{\mathfrak{L}}_{s_r\alpha} + g$, where g has β-degree less than $\mathsf{raj}(s_r\alpha) - |\alpha|$. Then

$$\begin{aligned}\mathfrak{L}_\alpha &= \pi_i(\mathfrak{L}_{s_r\alpha}) + \beta\pi_i(x_{i+1}\mathfrak{L}_{s_r\alpha})\\ &= \pi_i(\mathfrak{L}_{s_r\alpha}) + \beta\pi_i(x_{i+1}g) + \beta^{\mathsf{raj}(\alpha)-|\alpha|}\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_r\alpha})\end{aligned}$$

The first two terms on the right-hand side have β degree less than $\mathsf{raj}(\alpha) - |\alpha|$. Thus, the β-degree in $\mathfrak{L}_\alpha$ is at most $\mathsf{raj}(\alpha) - |\alpha|$. By Lemma 4.2, the β-degree in $\mathfrak{L}_\alpha$ is $\mathsf{raj}(\alpha) - |\alpha|$. Extract the coefficient of $\beta^{\mathsf{raj}(\alpha)-|\alpha|}$ and get

$$\widehat{\mathfrak{L}}_\alpha = \pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_r\alpha}) = c\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_r\gamma}) = c\widehat{\mathfrak{L}}_\gamma,$$

by Lemma 4.23. □

5. Snow diagrams for Rothe diagrams

Fix an $n \in \mathbb{Z}_{>0}$ throughout this section. We move on to study the snow diagrams of $RD(w)$ for $w \in S_n$. In subsection 5.1, we recall a version of Schensted insertion on S_n. In subection 5.2, we show the positions of dark clouds in $\mathsf{snow}(RD(w))$ is related to the Schensted insertion. We then use this connection to prove that $\mathsf{rajcode}(RD(w))$ is consistent to the $\mathsf{rajcode}(w)$ defined in [18]. In Section 5.3, we show the dark clouds in $\mathsf{snow}(RD(w))$ corresponds to the turning points in the shadow diagram for w. In Section 5.4, we study the snow diagrams for inverse fireworks permutations.

5.1. The Schensted Insertion.

If a diagram is top-justified and left-justified, we say it is a *Young diagram*. A filling of a Young diagram with positive integers is called a *tableau*. A tableau is called *partial* if it contains distinct numbers and each row (resp. column) is decreasing from left to right (resp. top to bottom). Notice that usually in literature, columns and rows are increasing. We reverse the convention to make our results easier to state.

The Schensted insertion [23] is an algorithm defined on a partial tableau T and a positive number x that is not in T. It finds the largest x' in the first row of T such that $x > x'$.

- If such x' does not exist, it appends x at the end of row one and terminates.
- Otherwise, it replaces x' by x and insert x' to the next row in the same way.

When the algorithm terminates, the resulting partial tableau is the output.

For $w \in S_n$, we insert $w(n), w(n-1), \dots, w(1)$ to the empty tableau via the Schensted insertion and denote the result by $P(w)$.

Example 5.1. Take $w \in S_7$ with one-line notation 3721564. The Schensted insertion on w yields:

$$\begin{array}{|c|}\hline 4\\\hline\end{array} \to \begin{array}{|c|}\hline 6\\\hline 4\\\hline\end{array} \to \begin{array}{|c|c|}\hline 6&5\\\hline 4\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 6&5&1\\\hline 4\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 6&5&2\\\hline 4&1\\\cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 7&5&2\\\hline 6&1\\\cline{1-2} 4\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 7&5&3\\\hline 6&2\\\cline{1-2} 4&1\\\cline{1-2}\end{array}.$$

One classical application of the Schensted insertion is to study increasing subsequences in a permutation. Recall $\mathrm{LIS}^w(q)$ is the length of the longest increasing subsequence of $w \in S_n$ that starts with q. It is related to the Schensted insertion as follows.

Lemma 5.2. [22, Lemma 3.3.3] *Take $w \in S_n$ and perform the Schensted insertion on w. For any $r \in [n]$, when $w(r)$ is inserted, it goes to column $\mathrm{LIS}^w(w(r))$ in row one.*

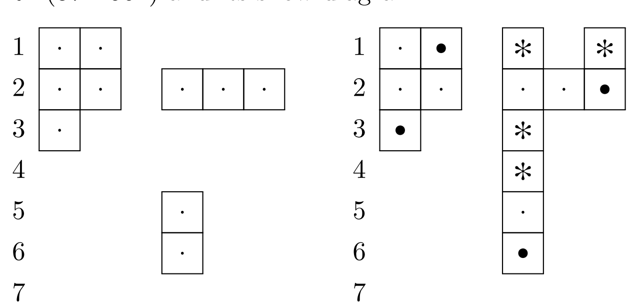
Example 5.3. Consider the $w \in S_7$ in Example 5.1. Notice that $\mathrm{LIS}^w(w(4)) = 3$. When $w(4) = 1$ is inserted to row one, it indeed goes to column 3.

5.2. Rajcode of Rothe diagrams. We show that $\mathsf{rajcode}(w)$ defined by Pechenik, Speyer and Weigandt (see Definition 2.1) agrees with the $\mathsf{rajcode}(RD(w))$ (see Definition 3.5). To do so, we need a better understanding of $\mathsf{snow}(RD(w))$. We start by describing how the positions of dark clouds in $\mathsf{snow}(RD(w))$ are related to the Schensted insertion described in Subsection 5.1.

Proposition 5.4. *Take $w \in S_n$. Consider the Schensted insertion on w. The dark cloud in row r of $\mathsf{snow}(RD(w))$ can be described based on the insertion of $w(r)$.*

(1) *If $w(r)$ is appended to the end of row one, then there is no dark cloud in the r^{th} row of $\mathsf{snow}(RD(w))$;*
(2) *If $w(r)$ bumps c in row one, then (r, c) is a dark cloud in $\mathsf{snow}(RD(w))$.*

Example 5.5. Let $w \in S_7$ with one-line notation 3721564. Consider the corresponding Rothe diagram $RD(3721564)$ and its snow diagram:



The Schensted insertion of w is presented in Example 5.1. We check Proposition 5.4 in the table below.

r	$w(r)$	insertion of $w(r)$ in row one	position of $\bullet$ in $\mathsf{snow}(RD(w))_r$
7	4	appended at the end of row one	row 7 has no $\bullet$
6	6	bumps 4 in row one	row 6 has $\bullet$ at $(6,4)$
5	5	appended at the end of row one	row 5 has no dark cloud
4	1	appended at the end of row one	row 4 has no dark cloud
3	2	bumps 1 in row one	row 3 has $\bullet$ at $(3,1)$
2	7	bumps 6 in row one	row 2 has $\bullet$ at $(2,6)$
1	3	bumps 2 in row one	row 1 has $\bullet$ at $(1,2)$

Proof. We prove the statement by induction on r starting from $r = n$. The number $w(n)$ is inserted into the empty tableau. In this case, it is appended to the end of the first row. It is also clear that there can not be any dark cloud on row n of $\mathsf{snow}(RD(w))$.

Now suppose the statement holds for $r+1, r+2, \ldots, n$ for some $r \leqslant n-1$. Let P be the tableau right before the insertion of $w(r)$. By the inductive hypothesis, for each $r' > r$, $w(r')$ appears in row 1 of P if and only if there is no dark cloud in column $w(r')$ under row r of $\mathsf{snow}(RD(w))$. Now consider the insertion of $w(r)$.

(1) Case 1: $w(r)$ is appended to the end of row 1.

Assume toward contradiction that $(r, w(r'))$ is a dark cloud of $\mathsf{snow}(RD(w))$ for some $r' > r$. Then $w(r) > w(r')$. Moreover, there is no dark cloud in the column of $w(r')$ under row r, so $w(r')$ is in row 1 of P. Thus, $w(r)$ cannot be appended in row 1, a contradiction.

(2) Case 2: $w(r)$ bumps $w(r')$ in row 1 for some $r' > r$.

Then $w(r) > w(r')$. The cell $(r, w(r'))$ is in $RD(w)$. We need to show that it is a dark cloud in $\mathsf{snow}(RD(w))$. By Remark 3.4, we just need to make sure there is no dark cloud under it or on its right.

Suppose that there is a dark cloud in column $w(r')$ under row r. By the inductive hypothesis, $w(r')$ cannot appear in row 1 of P, which is a contradiction.

Finally, suppose there is a dark cloud on the right of $(r, w(r'))$. We may write this dark cloud as $(r, w(r''))$ with $w(r'') > w(r')$. Since it is a cell in $RD(w)$, we also have $r'' > r$ and $w(r) > w(r'')$. Since it is a dark cloud, there is no dark cloud under it. By the inductive hypothesis, $w(r'')$ is in row 1 of P. This is a contradiction: $w(r)$ should bump $w(r'')$ instead of $w(r')$ since $w(r) > w(r'') > w(r')$. ☐

Theorem 5.6. *For $w \in S_n$,* $\mathsf{rajcode}(w) = \mathsf{rajcode}(RD(w))$.

Proof. Take $r \in [n]$. Consider row r of $\mathsf{snow}(RD(w))$. It contains $\mathsf{invcode}(w)_r$ cells that are not snowflakes. Let d_r be the number of dark clouds in $\mathsf{snow}(RD(w))$ that are southeast of $(r, w(r))$. Clearly, d_r is also the number of snowflakes in row r of $\mathsf{snow}(RD(w))$. We have $\mathsf{rajcode}(RD(w))_r = \mathsf{invcode}(w)_r + d_r$.

Consider the Schensted insertion of w. Let P be the tableau right before the insertion of $w(r)$. Define A as the number of elements in P that are larger than $w(r)$. We compute A in two ways.

- The tableau P consists of numbers $w(r+1), \ldots, w(n)$. There are $\mathsf{invcode}(w)_r$ of them less than $w(r)$, so $A = n - r - \mathsf{invcode}(w)_r$.
- Assume when inserting $w(r)$ to P, it goes to column c of row 1. Thus, $c-1$ is the number of entries in row 1 of P that are larger than $w(r)$. By Proposition 5.4, d_r is the number of entries under row 1 of P that are larger than $w(r)$. We have $A = c - 1 + d_r$. By Lemma 5.2, $c = \mathrm{LIS}^w(w(r))$, so $A = \mathrm{LIS}^w(w(r)) - 1 + d_r$.

Combining the two expressions of A yields

$$n - r - \mathsf{invcode}(w)_r = \mathrm{LIS}^w(w(r)) - 1 + d_r, \text{ so}$$

$$\mathsf{rajcode}(RD(w))_r = \mathsf{invcode}(w)_r + d_r = n - r + 1 - \mathrm{LIS}^w(w(r)) = \mathsf{rajcode}(w)_r.$$

☐

5.3. Dark Clouds of the Rothe Diagram via Viennot's geometric construction. In 1977, Xavier Gérard Viennot gave a diagrammatic construction of the RSK correspondence in terms of shadow lines ([26]). It is also known as the matrix-ball construction. We will show that the dark clouds in the snow diagram of a permutation can be obtained via Viennot's geometric construction. We denote $\mathsf{Row}_1(P(w))$ to be the first row of the tableau obtained by Schensted insertion on w.

For two cells $(i,j),(m,n) \in \mathbb{N}\times\mathbb{N}$, (m,n) lies in the shadow of (i,j) if and only if $m \leqslant i$ and $n \leqslant j$. This can be visualized by imagining shedding light from the Southeast.[(2)] To obtain the *shadow diagram* of $w \in S_n$, consider the points $(1,w(1)),\dots,(n,w(n))$. Let $\left(i_1^{(1)},w(i_1^{(1)})\right),\dots,\left(i_{\ell_1}^{(1)},w(i_{\ell_1}^{(1)})\right)$ be the points that are not in the shadow of any other point for some $\ell_1 \geqslant 1$ and $i_1^{(1)} > i_2^{(1)} > \cdots > i_{\ell_1}^{(1)}$. Then the first *shadow line* $L_1(w)$ is the boundary of the combined shadows of the points $\left(i_1^{(1)},w(i_1^{(1)})\right),\dots,\left(i_{\ell_1}^{(1)},w(i_{\ell_1}^{(1)})\right)$. The rest of the $L_j(w)$ can be constructed recursively. Supposed $L_1,\dots,L_{j-1}$ have been constructed, remove all points in the set

$$\left\{\left(i_k^{(p)},w(i_k^{(p)})\right) : 1 \leqslant p \leqslant j-1, 1 \leqslant k \leqslant \ell_p\right\},$$

then L_j is the boundary of the shadow of the remaining points of the points left, which we label as

$$\left(i_1^{(j)},w(i_1^{(j)})\right),\dots\left(i_{\ell_j}^{(j)},w(i_{\ell_j}^{(j)})\right),$$

for some $\ell_j \geqslant 1$ and $i_1^{(j)} > i_2^{(j)} > \cdots > i_{\ell_j}^{(j)}$. Once there is no point left, the shadow lines we obtained form the shadow diagram for w.

Theorem 5.7 ([26]). *Given $w \in S_n$ and suppose $L_1,\dots,L_s$ are the shadow lines obtained from w until there is no point left. Then s equals the size of $\mathsf{Row}_1(P(w))$.*

For each shadow line L_j, it also consists $\ell_j - 1$ "turning points", which are points (x,y) of L_j such that $(x-1,y),(x,y-1) \notin L_j$, i.e.,

$$\left(i_2^{(j)},w(i_1^{(j)})\right),\left(i_3^{(j)},w(i_2^{(j)})\right),\dots,\left(i_{\ell_j}^{(j)},w(i_{\ell_j-1}^{(j)})\right).$$

In total, there are $n-|\mathsf{Row}_1(P(w))|$ turning points for each $w \in S_n$. There is a classical result connecting these turning points to the Schensted insertion.

Theorem 5.8 ([26, 10]). *Let a shadow line L_j of a permutation w consists of points*

$$\left(i_1^{(j)},w(i_1^{(j)})\right),\dots\left(i_{\ell_j}^{(j)},w(i_{\ell_j}^{(j)})\right)$$

for some $\ell_j \geqslant 1$ and $i_1^{(j)} > i_2^{(j)} > \cdots > i_{\ell_j}^{(j)}$. Then during Schensted insertion on w, when we insert $w(i_{k+1}^{(j)})$, it bumps $w(i_k^{(j)})$ from the first row.

Combining Proposition 5.4 and Theorem 5.8, we have the following.

Corollary 5.9. *Each of the turning points in the shadow diagram of w contains a dark cloud in $\mathsf{snow}(w)$. Any dark cloud in $\mathsf{snow}(w)$ is also a turning point in the shadow diagram of w.*

Example 5.10. Consider $w = 3721564 \in S_7$. We present its Rothe diagram, its shadow diagram, and the snow diagram of $RD(w)$. From Example 5.1, the Schensted insertion on w yields a tableau whose row 1 has three cells. Correspondingly, there are three shadow lines. The turning points of the shadow lines are $(3,1),(1,2),(6,4),(2,6)$, which are positions for dark clouds in $\mathsf{snow}(RD(w))$.

[(2)] The usual convention can be thought of shedding light from the Northwest, which corresponds to the usual Schensted insertion. We reverse the direction to match our decreasing Schensted insertion convention.

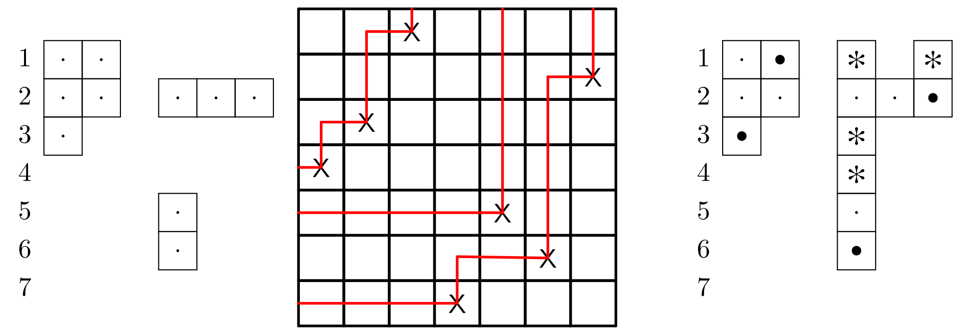


FIGURE 1. Left: $\mathrm{RD}(w)$; Middle: shadow diagram of w; Right: $\mathsf{snow}(RD(w))$

REMARK 5.11. A geometric interpretation for the rajcode is given in [18, Section 4] in terms of the "blob diagrams." Specifically, the set of points in the same shadow line in the shadow line diagram is labeled as $B_n, B_{n-1}, \dots$ from southeast to northwest. With the labeling on the blob diagrams, we can obtain the rajcode directly. That is, if $(i, w(i)) \in B_k$, then $\mathsf{rajcode}(w)_i = k - i$.

5.4. INVERSE FIREWORKS PERMUTATIONS. Now we have seen that our snow diagrams are connected to the work of Pechenik, Speyer and Weigandt [18]. We recall another interesting notion in their work.

DEFINITION 5.12 ([18, Definition 3.5]). *A permutation $w \in S_n$ is a fireworks permutation if its initial element in each decreasing run is increasing. A permutation $w \in S_n$ is an inverse fireworks permutation if w^{-1} is a fireworks permutation.*

Inverse fireworks permutations are the representatives of equivalence classes, given by permutations with the same $\mathsf{rajcode}$ [18]. The snowy weak compositions play the same role in our study of $\widehat{\mathfrak{L}}$. We investigate the similarities between inverse fireworks permutations and snowy weak compositions. For w inverse fireworks, $RD(w)$ enjoy analogous properties as the $D(\alpha)$ of snowy α. We start with the following observation about $RD(w)$.

LEMMA 5.13. *Let $w \in S_n$ be an inverse fireworks permutation. Consider each $r \in [n]$ such that row r of $RD(w)$ is not empty. The rightmost cell in row r of $RD(w)$ is $(r, w(r) - 1)$.*

Proof. Recall that $(r, w(r')) \in RD(w)$ if and only if $(r, r') \in \mathsf{Inv}(w)$ if and only if $(w(r'), w(r)) \in \mathsf{Inv}(w^{-1})$. Let $c = w(r)$. Clearly, cells in row r of $RD(w)$ are within the first $c - 1$ columns. It remains to check $(r, c-1) \in RD(w)$, which is equivalent to $(c-1, c) \in \mathsf{Inv}(w^{-1})$.

Since row r of $RD(w)$ is nonempty, it must contain a cell (r, i) such that $(i, c) \in \mathsf{Inv}(w^{-1})$ for some $i \in [c-1]$. Since $w^{-1}(i) > w^{-1}(c)$ and w^{-1} is fireworks, $w^{-1}(c)$ can not be the initial element in its decreasing run. Therefore $w^{-1}(c-1) > w^{-1}(c)$ and we have $(c-1, c) \in \mathsf{Inv}(w^{-1})$. $\square$

We can characterize the inverse fireworks permutations using Rothe diagrams or the snow diagram of the permutation. This is similar to Remark 4.17, where we describe snowy weak compositions using key diagrams and dark clouds.

PROPOSITION 5.14. *Take $w \in S_n$. The following are equivalent:*

(1) *w is an inverse fireworks permutation.*
(2) *In $RD(w)$, the rightmost cells in each row are in different columns.*
(3) *In $\mathsf{snow}(RD(w))$, the rightmost cell in each row is a dark cloud.*

Proof. The last two statements are clearly equivalent. Now we establish the equivalence of the first two statements.

Assume w is inverse fireworks. Take $r, r' \in [n]$ with $r \neq r'$ such that row r and row r' of $RD(w)$ are not empty. By Lemma 5.13, the rightmost cell in row r (resp. r') is at $(r, w(r)-1)$ (resp. $(r', w(r')-1)$). Clearly, $w(r)-1 \neq w(r')-1$, so we have our second statement.

Now we assume w is not inverse fireworks. We can find a number r in w^{-1} such that r is the initial element in its decreasing run, but r is less than r', the initial element of the previous decreasing run. Let $c' = w(r')$ and $c = w(r)$. Since $(c', c) \in \mathsf{Inv}(w^{-1})$, $(r, c') \in RD(w)$. Thus, row r of $RD(w)$ is not empty. Let (r, i) be the rightmost cell in row r. In other words, i is the largest such that $(i, c) \in \mathsf{Inv}(w^{-1})$. We have $c' \leqslant i < c-1$. Consider the decreasing run before $w^{-1}(c)$: $w^{-1}(c') > w^{-1}(c'+1) > \cdots > w^{-1}(c-1)$. We see $(i, i+1)$ is also in $\mathsf{Inv}(w^{-1})$. In row $w^{-1}(i+1)$, the cell $(w^{-1}(i+1), i)$ is the rightmost cell of its row. Thus, the second statement does not hold, and the proof is finished. ☐

With the above proposition, we can compute $\mathsf{rajcode}(w)$ easily if w is inverse fireworks. The following rule is similar to Lemma 4.18(2).

Proposition 5.15. *Assume $w \in S_n$ is inverse fireworks. For each $r \in [n]$,*

$$\mathsf{rajcode}(w)_r = |\{r' > r : (r, r') \in \mathsf{Inv}(w) \text{ or } w(r') > w(r) \text{ and } (r', r'') \in \mathsf{Inv}(w) \text{ for some } r''\}|.$$

Proof. First, we know $\mathsf{rajcode}(w)_r = \mathsf{rajcode}(RD(w))_r$ is the number of cells in the r^{th} row of $\mathsf{snow}(RD(w))$. The number of non-snowflake cells on this row is given by $|\{r' : (r, r') \in \mathsf{Inv}(w)\}|$.

Now we count the number of snowflakes in row r of $\mathsf{snow}(RD(w))$. It is the number of $r' > r$ such that row r' of $\mathsf{snow}(RD(w))$ has a dark cloud on the right of the column $w(r)$. By Lemma 5.13, row r' has a dark cloud at column $w(r')-1$ if $RD(w)$ is nonempty in row r'. Thus, the number of snowflakes in row r of $\mathsf{snow}(RD(w))$ is the number of $r' > r$ such that $w(r') > w(r)$ and $(r', r'') \in \mathsf{Inv}(w)$ for some r''. ☐

6. Vector space spanned by $\widehat{\mathfrak{G}}_w$

We now study the spaces $\widehat{V}_n := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{G}}_w : w \in S_n\}$ and $\widehat{V} := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{G}}_w : w \in S_+\}$. By Theorem 1.1, they have bases

$$\{\widehat{\mathfrak{G}}_w : w \in S_n \text{ is inverse fireworks}\} \quad \text{and} \quad \{\widehat{\mathfrak{G}}_w : w \in S_+ \text{ is inverse fireworks}\}$$

respectively. By [4], the number of inverse fireworks permutations in S_n is B_n, the n^{th} Bell number. Thus, $\widehat{V}_n$ has dimension B_n.

We introduce another basis of $\widehat{V}_n$ and $\widehat{V}$ consisting of $\widehat{\mathfrak{L}}_\alpha$, the top-degree components of Lascoux polynomials. One application of the top Lascoux basis is to compute the Hilbert series of $\widehat{V}_n$ and $\widehat{V}$. For a vector space $V \subseteq \mathbb{Q}[x_1, x_2, \cdots]$, the *Hilbert series* of V is

$$\mathrm{Hilb}(V; q) := \sum_{d \geqslant 0} m_d q^d,$$

where m_d is the number of polynomials with degree d in a homogeneous basis of V.

In Subsection 6.1, we recall the definition of B_n and its q-analogue $B_n(q)$. In Subsection 6.2, we compute $\mathrm{Hilb}(\widehat{V}_n; q)$ using $B_n(q)$ and rook-theoretic results. In Subsection 6.3, we compute $\mathrm{Hilb}(\widehat{V}; q)$.

6.1. Stirling numbers, Bell numbers and their q-analogues. Let n, k be non-negative integers throughout this subsection. Let $S_{n,k}$ be the *Stirling number of the second kind*, defined by the recurrence relation

$$S_{n+1,k} = S_{n,k-1} + kS_{n,k},$$

together with $S_{0,0} = 1$ and $S_{0,k} = 0$ if $k > 0$. Let $B_n := \sum_{j=0}^{n} S_{n,j}$ be the Bell number which satisfies the following recurrence relation

$$B_{n+1} = \sum_{j=0}^{n} \binom{n}{j} B_j.$$

Let Rook_n be the set of non-attacking rook diagrams contained in Stair_n. It is an exercise to show $B_n = |\mathsf{Rook}_n|$. In [3], Butler, Can, Haglund, and Remmel built an explicit bijection between Rook_n and set partitions of $[n]$.

Now consider the polynomial ring $\mathbb{Q}[q]$. Define $[n]_q := 1 + q + \cdots + q^{n-1}$. Define a *$q$-analogue of $S_{n,k}$* recursively by:

$$S_{n+1,k}(q) = q^{k-1} S_{n,k-1}(q) + [k]_q S_{n,k}(q),$$

with base cases $S_{0,k}(q) = S_{0,k}$. Similarly, define a *q-analogue of B_n* by $B_n(q) := \sum_{j=0}^{n} S_{n,j}(q)$. The coefficients in $B_n(q)$ are given in OEIS A126347. By [27], $B_n(q)$ satisfies the recurrence relation

$$B_{n+1}(q) = \sum_{j=0}^{n} q^j \binom{n}{j} B_j(q).$$

Milne [16] first gave a combinatorial model for $S_{n,k}(q)$ using set partitions. We use the combinatorial model developed by Garsia and Remmel [8]. They defined a statistic on Rook_n called "inversion". We rename it as GR_n to distinguish it from the inversion on permutations.

Definition 6.1 ([8]). *Assume $R \in \mathsf{Rook}_n$. For each $(r, c) \in R$, mark all cells (r', c) with $r' \in [r]$ in Stair_n. Also, mark all cells (r, c') with $c' \in [c]$ in Stair_n. The number $\mathsf{GR}_n(R)$ counts cells in Stair_n that are not marked.*

Garsia and Remmel prove that

$$S_{n,k}(q) = \sum_{\substack{D \in \mathsf{Rook}_n \\ |D| = n-k}} q^{\mathsf{GR}_n(D)}, \tag{5}$$

which implies

$$B_n(q) = \sum_{D \in \mathsf{Rook}_n} q^{\mathsf{GR}_n(D)}. \tag{6}$$

From this formula, $B_n(q)$ has degree $\binom{n}{2}$ since $\mathsf{GR}_n(\varnothing) = \binom{n}{2}$.

6.2. Computing $\mathrm{Hilb}(\widehat{V}_n; q)$. Define

$$C_n := \{\alpha \in C_+ : \mathsf{supp}(\alpha) \subseteq [n-1], \alpha_i \leqslant n - i \text{ for all } i \in [n-1]\}.$$

Then we can refine Theorem 2.4.

Corollary 6.2. *For $w \in S_n$, $\mathfrak{G}_w$ expands positively into $\{\mathfrak{L}_\alpha : \alpha \in C_n\}$.*

Proof. By Theorem 2.4, we can expand $\mathfrak{G}_w$ into a sum of Lascoux polynomials. We just need to make sure for each $\mathfrak{L}_\alpha$ appearing in the expansion with a nonzero coefficient, we have $\alpha \in C_n$.

We know the monomial x^α is the leading monomial of κ_α, so x^α appears in $\mathfrak{L}_\alpha$. Since all coefficients in the sum are positive, we know $x^\alpha \beta^m$ appears in $\mathfrak{G}_w$ for some

$m \in \mathbb{Z}_{\geqslant 0}$. By the monomial expansion of $\mathfrak{G}_w$ given by Fomin and Kirillov [7], we have $\alpha \in C_n$. $\square$

By this corollary and Lemma 2.5, we have the following.

Corollary 6.3. *For $w \in S_n$, $\widehat{\mathfrak{G}}_w$ expands positively into $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$.*

Now we are ready to give another basis of $\widehat{V}_n$:

Proposition 6.4. *The space $\widehat{V}_n$ is also $\mathbb{Q}$-span$\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$. Moreover, it has a basis $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n$ is snowy$\}$.*

Proof. By Corollary 6.3 , $\widehat{V}_n$ is a subspace of $\mathbb{Q}$-span$\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$. By Lemma 4.5, for any $\alpha \in C_n$, we can find a snowy $\gamma \in C_n$ such that $\gamma \sim \alpha$. Then by Theorem 1.2, $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{L}}_\gamma$. Thus, $\mathbb{Q}$-span$\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$ is a subspace of the vector space $\mathbb{Q}$-span$\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n$ is snowy$\}$. Notice that $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n$ is snowy$\}$ is linear independent since its polynomials have distinct leading terms by Theorem 1.2.

By [18, Theorem 1.4] $\widehat{V}_n$ has dimension B_n. It remains to check the number of snowy weak compositions in C_n is also B_n. In Lemma 4.20, we show $\mathsf{dark}(\cdot)$ a bijection from snowy weak compositions in C_+ to Rook_+. Clearly it restricts to a bijection from C_n to Rook_n, which has size B_n. $\square$

We use the top Lascoux basis to derive $\mathrm{Hilb}(\widehat{V}_n; q)$. Let us translate the statistic $\mathsf{raj}(\cdot)$ on snowy weak compositions to non-attacking rook diagrams.

Definition 6.5. *Take $R \in \mathsf{Rook}_+$. Define the Northwest number of R, denoted as $\mathsf{NW}(R) := \mathsf{raj}(\alpha)$, where α is any weak composition with $\mathsf{dark}(\alpha) = R$.*

Equivalently, we may compute $\mathsf{NW}(R)$ as follows: For each $(r, c) \in R$, we mark all cells weakly above it and to its left. By Lemma 4.14, these marked cells agree with the underlying diagram of $\mathsf{snow}(D(\alpha))$ for any α with $\mathsf{dark}(\alpha) = R$. Then $\mathsf{NW}(R)$ is just the number of marked cells. Comparing this statistic with $\mathsf{GR}_n(\cdot)$ defined in Subsection 6.1, we have the following connection.

Remark 6.6. Take $R \in \mathsf{Rook}_n$. Then $\mathsf{GR}_n(R) = |\mathsf{Stair}_n| - \mathsf{NW}(R) = \binom{n}{2} - \mathsf{NW}(R)$.

Finally, we can derive an expression for the degree generating function of $\widehat{V}_n$.

Proposition 6.7. *We have*

$$\mathrm{Hilb}(\widehat{V}_n; q) = q^{\binom{n}{2}} B_n(q^{-1}) = \mathsf{rev}(B_n(q)),$$

where $\mathsf{rev}(\cdot)$ is the operator that reverse the coefficients of a polynomial. In other words, it sends a polynomial $f(q)$ of degree d to $q^d f(q^{-1})$.

Proof. By Prop 6.4, $\mathrm{Hilb}(\widehat{V}_n; q) = \sum_\alpha q^{\mathsf{raj}(\alpha)}$ where the sum is over snowy $\alpha \in C_n$. Apply the bijection $\mathsf{dark}(\cdot)$ to α in the summation, we have

$$\begin{aligned}\mathrm{Hilb}(\widehat{V}_n; q) &= \sum_{R \in \mathsf{Rook}_n} q^{\mathsf{NW}(R)} = \sum_{R \in \mathsf{Rook}_n} q^{\binom{n}{2} - \mathsf{GR}_n(R)} \\ &= q^{\binom{n}{2}} \sum_{R \in \mathsf{Rook}_n} q^{-\mathsf{GR}_n(R)} = q^{\binom{n}{2}} B_n(q^{-1}),\end{aligned}$$

where the second equality is by Remark 6.6 and the last equality is by (6). Since $B_n(q)$ has degree $\binom{n}{2}$, we have $\mathrm{Hilb}(\widehat{V}_n; q) = \mathsf{rev}(B_n(q))$. $\square$

6.3. Computing $\mathrm{Hilb}(\widehat{V};q)$. First, we show the top Lascoux polynomials also span the space $\widehat{V}$.

Proposition 6.8. *We have* $\mathbb{Q}$-*span*$\{\widehat{\mathfrak{L}}_\alpha \mid \alpha \in C_+\} = \widehat{V}$.

Proof. By Corollary 2.6, $\widehat{V}$ is in the $\mathbb{Q}$-span of $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_+\}$. Now consider $\alpha \in C_+$. There exists n large enough such that $\alpha \in C_n$. Then $\widehat{\mathfrak{L}}_\alpha \in \widehat{V}_n \subset \widehat{V}$. □

Corollary 6.9. *The space* $\widehat{V}$ *has a basis* $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_+$ *is snowy*$\}$.

With the top Lascoux basis, we have

$$\mathrm{Hilb}(\widehat{V};q) \quad = \sum_{\substack{\alpha \in C_+,\\ \alpha \text{ is snowy}}} q^{\mathsf{raj}(\alpha)} \quad = \sum_{R \in \mathsf{Rook}_+} q^{\mathsf{NW}(R)},$$

where the second equality is obtained by applying $\mathsf{dark}(\cdot)$ on α in the second expression. On the other hand, since $\widehat{V} = \bigcup_{n \geqslant 1} \widehat{V}_n$,

$\mathrm{Hilb}(\widehat{V};q)$ is the limit of $\mathrm{Hilb}(\widehat{V}_n;q)$ as n goes to infinity. According to OEIS, coefficients in $B_n(q)$ are in A126347 and the coefficients of $\mathrm{Hilb}(\widehat{V};q)$ are in A126348. A formula for $\mathrm{Hilb}(\widehat{V};q)$ in OEIS is given by Jovovic: $\prod_{m>0}(1+\frac{q^m}{1-q})$. For completeness, we check this rule using our formula of $\mathrm{Hilb}(\widehat{V};q)$ involving snowy weak compositions.

Proposition 6.10. *We have*

$$\mathrm{Hilb}(\widehat{V};q) = \sum_{\alpha \text{ is snowy}} q^{\mathsf{raj}(\alpha)} = \prod_{m>0}\left(1+\frac{q^m}{1-q}\right)$$

Proof. Let $\mathsf{snowy}(M)$ be the set of all snowy weak compositions with the largest entry being at most M. It suffices to show

$$\sum_{\alpha \in \mathsf{snowy}(M)} q^{\mathsf{raj}(\alpha)} = \prod_{m>0}^{M}\left(1+\frac{q^m}{1-q}\right).$$

We prove it by induction on M. The claim is immediate when $M=0$ as both sides are 1.

Now assume the claim above holds for some $M \geqslant 0$. Let $\mathsf{snowy}(M)_i$ be the set of all snowy weak compositions α such that its largest entry is $\alpha_i = M$. With this notation, we can express $\mathsf{snowy}(M)$ recursively:

$$\mathsf{snowy}(M) = \mathsf{snowy}(M-1) \bigsqcup \left(\bigsqcup_{i \geqslant 1} \mathsf{snowy}(M)_i\right).$$

Next, we define a map

$$\begin{aligned} \phi : \mathsf{snowy}(M-1) &\to \mathsf{snowy}(M)_1 \\ (\alpha_1, \alpha_2, \dots) &\mapsto (M, \alpha_1, \alpha_2, \dots) \end{aligned}$$

It is straightforward to see that ϕ is a bijection. Furthermore, we have $\mathsf{raj}(\phi(\alpha)) = \mathsf{raj}(\alpha) + M$. To get $\mathsf{snowy}(M)_i$ for $i > 1$, notice that the operator s_i on the set of weak compositions is a bijection between $\mathsf{snowy}(M)_i$ and $\mathsf{snowy}(M)_{i+1}$. For $\alpha \in \mathsf{snowy}(M)_i$, we have $\mathsf{raj}(s_i(\alpha)) = \mathsf{raj}(\alpha) + 1$ by Corollary 4.19. Inductively, we have

$$\sum_{\alpha \in \mathsf{snowy}(M)_i} q^{\mathsf{raj}(\alpha)} = q^{M+i-1} \sum_{\alpha \in \mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)}.$$

Finally,

$$\begin{aligned}\sum_{\alpha\in\mathsf{snowy}(M)} q^{\mathsf{raj}(\alpha)} &= \sum_{\alpha\in\mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)} + \left(\sum_{i\geqslant 1} q^{M+i-1}\right) \sum_{\alpha\in\mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)} \\ &= \left(1+\sum_{i\geqslant 1} q^{M+i-1}\right) \sum_{\alpha\in\mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)} \\ &= \left(1+\frac{q^M}{1-q}\right)\prod_{m>0}^{M-1}\left(1+\frac{q^m}{1-q}\right).\end{aligned}$$

□

7. Open Problems and Future Directions

We conclude with several open problems for future study. In Section 5.3, we present the connections between the following three constructions:

- Positions of dark clouds in $\mathsf{snow}(RD(w))$;
- First step of Viennot's geometric construction;
- Bumps in the first row during Schensted insertion.

Question 7.1. *Find further connections between Viennot's geometric construction of Schensted insertion and* $\mathsf{snow}(RD(w))$.

Question 7.2. *Find further connections between the Schensted insertion and the snow diagram of a permutation.*

The Grothendieck to Lascoux expansion, proven in [24], involves finding certain tableaux and computing their right keys.

Question 7.3. *Find a combinatorial formula for the expansion of Castelnuovo–Mumford polynomials into top Lascoux polynomials indexed by snowy weak compositions.*

Finding a combinatorial formula for the *structure constants* $c^w_{u,v}$ for Grothendieck polynomials, defined as

$$\mathfrak{G}_u\mathfrak{G}_v = \sum_w c^w_{u,v}\mathfrak{G}_w\,,$$

has been a long-standing open problem. These coefficients have a geometric interpretation: They are the intersection numbers for the Schubert classes in the connective K-theory. If we consider only the top-degree terms on both sides, we get the structure constants for Castelnuovo–Mumford polynomials, which we denote as $\widehat{c^w_{uv}}$, which are still non-negative integers.

Question 7.4. *Find a combinatorial formula for* $\widehat{c^w_{uv}}$.

The Grothendieck polynomial $\mathfrak{G}_w(\boldsymbol{x})$ is a specialization of the double Grothendieck polynomial $\mathfrak{G}_w(\boldsymbol{x},\boldsymbol{y})$ by setting $y_1 = y_2 = \cdots = 0$. In [11], Knutson and Miller introduced pipe dream rules for both $\mathfrak{G}_w(\boldsymbol{x})$ and $\mathfrak{G}_w(\boldsymbol{x},\boldsymbol{y})$. For Castelnuovo–Mumford polynomials $\widehat{\mathfrak{G}_w}(\boldsymbol{x})$, we can think they correspond to a subset of pipe dreams for $\mathfrak{G}_w(\boldsymbol{x})$. In [18], the authors proved a factorization of $\widehat{\mathfrak{G}_w}(\boldsymbol{x},\boldsymbol{y})$ into a $\boldsymbol{x}$-polynomial and a $\boldsymbol{y}$-polynomial, and they showed the the leading term is in fact,

$$\boldsymbol{x}^{\mathsf{rajcode}(w)}\boldsymbol{y}^{\mathsf{rajcode}(w^{-1})},$$

with coefficient 1 by constructing a pipe dream associated with it iteratively.

Question 7.5. *Use the snow diagrams to give an explicit construction of pipe dreams for the leading term in* $\widehat{\mathfrak{G}_w}(\boldsymbol{x},\boldsymbol{y})$.

In general, one can define a K-Kohnert polynomial for any diagram D:

$$\kappa_D(\boldsymbol{x};\beta) := \sum_{D' \in \mathsf{KKD}(D)} \boldsymbol{x}^{\mathsf{wt}(D')}\beta^{\mathsf{ex}(D')}.$$

Question 7.6. *Find characterizations of diagram* D *such that the leading monomial of* $\widehat{\kappa}_D$ *is given by* $\mathsf{rajcode}(D)$.

8. Appendix

Permutation w	$\mathfrak{G}_w$ ($\widehat{\mathfrak{G}_w}$ in bold blue)
$e^\dagger$	$\boldsymbol{1}$
$2134^\dagger = s_1$	$\boldsymbol{x_1}$
$1324^\dagger = s_2$	$(x_1 + x_2) + \beta\boldsymbol{x_1x_2}$
$2314 = s_1s_2$	$\boldsymbol{x_1x_2}$
$3124^\dagger = s_2s_1$	$\boldsymbol{x_1^2}$
$3214^\dagger = s_1s_2s_1$	$\boldsymbol{x_1^2x_2}$
$1243^\dagger = s_3$	$(x_1 + x_2 + x_3) + \beta(x_1x_2 + x_1x_3 + x_2x_3) + \beta^2\boldsymbol{x_1x_2x_3}$
$2143^\dagger = s_1s_3$	$(x_1x_2 + x_1x_3 + x_1^2) + \beta(x_1x_2x_3 + x_1^2x_2 + x_1^2x_3)$ $+\beta^2\boldsymbol{x_1^2x_2x_3}$
$1342 = s_2s_3$	$(x_1x_2 + x_1x_3 + x_2x_3) + \beta\boldsymbol{2x_1x_2x_3}$
$1423^\dagger = s_3s_2$	$(x_1^2 + x_2^2 + x_1x_2) + \beta(\boldsymbol{x_1^2x_2 + x_1x_2^2})$
$2341 = s_1s_2s_3$	$\boldsymbol{x_1x_2x_3}$
$2413^\dagger = s_1s_3s_2$	$(x_1x_2^2 + x_1^2x_2) + \beta\boldsymbol{x_1^2x_2^2}$
$3142 = s_2s_1s_3$	$(x_1^2x_2 + x_1^2x_3) + \beta\boldsymbol{x_1^2x_2x_3}$
$4123^\dagger = s_3s_2s_1$	$\boldsymbol{x_1^3}$
$1432^\dagger = s_3s_2s_3$	$(x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3)$ $+ \beta(x_1^2x_2^2 + 2x_1^2x_2x_3 + 2x_1x_2^2x_3) + \beta^2\ \boldsymbol{x_1^2x_2^2x_3}$
$2431 = s_1s_3s_2s_3$	$(x_1^2x_2x_3 + x_1x_2^2x_3) + \beta\boldsymbol{x_1^2x_2^2x_3}$
$3241 = s_2s_1s_2s_3$	$\boldsymbol{x_1^2x_2x_3}$
$3412 = s_2s_1s_3s_2$	$\boldsymbol{x_1^2x_2^2}$
$4132^\dagger = s_3s_2s_1s_3$	$(x_1^3x_2 + x_1^3x_3) + \beta\boldsymbol{x_1^3x_2x_3}$
$4213^\dagger = s_3s_2s_1s_2$	$\boldsymbol{x_1^3x_2}$
$3421 = s_2s_1s_3s_2s_3$	$\boldsymbol{x_1^2x_2^2x_3}$
$4231 = s_3s_2s_1s_2s_3$	$\boldsymbol{x_1^3x_2x_3}$
$4312^\dagger = s_3s_2s_1s_3s_2$	$\boldsymbol{x_1^3x_2^2}$
$4321^\dagger = s_3s_2s_1s_3s_2s_3$	$\boldsymbol{x_1^3x_2^2x_3}$

Table 1. Grothendieck polynomials in S_4. † refers to inverse fireworks permutations.

Weak compositions α	$\mathfrak{L}_\alpha$ ($\widehat{\mathfrak{L}}_\alpha$ in bold blue)
$(0,0,0)^\dagger$	$\boldsymbol{1}$
$(1,0,0)^\dagger$	$\boldsymbol{x_1}$
$(0,1,0)^\dagger$	$(x_1+x_2)+\beta\ \boldsymbol{x_1x_2}$
$(1,1,0)$	$\boldsymbol{x_1x_2}$
$(2,0,0)^\dagger$	$\boldsymbol{x_1^2}$
$(2,1,0)^\dagger$	$\boldsymbol{x_1^2x_2}$
$(0,0,1)^\dagger$	$(x_1+x_2+x_3)+\beta(x_1x_2+x_1x_3+x_2x_3)+\beta^2\boldsymbol{x_1x_2x_3}$
$(0,1,1)$	$(x_1x_2+x_1x_3+x_2x_3)+\beta\boldsymbol{2x_1x_2x_3}$
$(0,2,0)^\dagger$	$(x_1^2+x_1x_2+x_2^2)+\beta\ (\boldsymbol{x_1^2x_2+x_1x_2^2})$
$(1,0,1)$	$(x_1x_2+x_1x_3)+\beta\boldsymbol{x_1x_2x_3}$
$(3,0,0)^\dagger$	$\boldsymbol{x_1^3}$
$(2,0,1)^\dagger$	$(x_1^2x_2+x_1^2x_3)+\beta\boldsymbol{x_1^2x_2x_3}$
$(1,2,0)^\dagger$	$(x_1x_2^2+x_1^2x_2)+\beta\boldsymbol{x_1^2x_2^2}$
$(0,2,1)^\dagger$	$(x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_2x_3+x_2^2x_3)$
	$+\beta(x_1^2x_2^2+2x_1^2x_2x_3+2x_1x_2^2x_3)+\beta^2\boldsymbol{x_1^2x_2^2x_3}$
$(1,1,1)$	$\boldsymbol{x_1x_2x_3}$
$(3,1,0)^\dagger$	$\boldsymbol{x_1^3x_2}$
$(3,0,1)^\dagger$	$(x_1^3x_2+x_1^3x_3)+\beta\boldsymbol{x_1^3x_2x_3}$
$(2,2,0)$	$\boldsymbol{x_1^2x_2^2}$
$(2,1,1)$	$\boldsymbol{x_1^2x_2^2}$
$(1,2,1)$	$(x_1^2x_2x_3+x_1x_2^2x_3)+\beta\boldsymbol{x_1^2x_2^2x_3}$
$(3,2,0)^\dagger$	$\boldsymbol{x_1^3x_2^2}$
$(3,1,1)$	$\boldsymbol{x_1^3x_2x_3}$
$(2,2,1)$	$\boldsymbol{x_1^2x_2^2x_3}$
$(3,2,1)^\dagger$	$\boldsymbol{x_1^3x_2^2x_3}$

Table 2. Lascoux polynomials in C_4. † refers to snowy weak compositions.

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